A Boson Representation for SU (N) Lattice Gauge Theories

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Abstract. SU(N) lattice gauge theories are reformulated in terms of fields varying over non-compact spaces \( \mathbb{C}^N \), transforming as \( N \) dimensional representations of SU(N) and integrated with Gaussian measure. This reformulation is equivalent to a boson operator representation. Strong coupling expansions based on this formalism do not involve SU(N) vector coupling coefficients.

1. Introduction

In pure Euclidean Yang–Mills field theories on a lattice field, variables range over the group manifold itself. This manifold is compact and a non-trivial Riemannian space. The gauge groups we will consider are SU(N), \( N = 2, 3 \) but our results can be immediately generalized to any \( N \). In this article we reformulate such theories in an equivalent fashion in terms of fields taken from the flat non-compact space \( \mathbb{C}^N \). They transform as \( N \) dimensional representations of SU(N). We will therefore call them “bosonic spinorial variables” for the gauge field. The integration is over a Gaussian measure instead of a Haar measure. A straightforward change of notation leads then to a boson operator formulation of Yang–Mills lattice field theories.

Our approach is based on Bargmann’s realization of group representations of SU(N) [1], which makes use of Hilbert spaces of entire analytic functions over \( \mathbb{C}^N \) or powers of \( \mathbb{C}^N \). This formalism is equivalent to the so-called boson operator calculus [2]. For technical reasons and for the sake of mathematical clarity we prefer to use spaces of analytic functions in this article.

The lattice \( A \) is assumed to be hypercubic, to have dimension \( D \) and the boundary conditions are presumed to be periodic. Let \( \ell \) denote the links and \( p \) the plaquettes of \( A \). We define the partition function by the standard ansatz

\[
Z = \int \prod_{\ell \in A} (du_\ell) e^{S(u_\ell)}.
\]

\[ u_\ell \in \text{SU}(N) \tag{1} \]

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The action $S$ can be represented as a sum of contributions $S_p$ of each plaquette $p$

$$S\{\{u_p\}\} = \sum_{\text{plaquettes } p \in A} S_p.$$  

(2)

We assume that each $S_p$ can be expanded into characters

$$e^{S_p} = \sum_{\{R\}} \dim \chi^R(u_{\partial p}) f_R(\beta).$$

(3)

Here $\{R\}$ is the set of irreducible unitary representations of SU($N$), $\chi^R$ are the corresponding characters. $\beta$ is the inverse temperature or the inverse coupling constant squared and $f_R(\beta)$ are "dynamical factors" that specify the action. The Wilson action [3] or the generalized Villain action [4] are included as special cases. The argument $u_{\partial p} \in$SU($N$) is the usual product of group elements $u_p$ along the boundary $\partial p$ of $p$.

For the purpose of generality we use (3) as a starting point. After the introduction of bosonic spinorial variables all integrations over gauge group variables can be performed exactly. If we are able to sum over $R$ for certain given functions $f_R(\beta)$, we can study the reformulated Yang–Mills theory both for $\beta \to 0$ (the strong coupling limit) as for $\beta \to \infty$ (the weak coupling limit). Without this summation over $R$ the reformulated Yang–Mills theory can only be studied in the strong coupling domain. Since all integrations in the strong coupling expansion are now Gaussian, they are elementary. Vector coupling coefficients of SU($N$) do not arise. This is of particular interest, since for $N \geq 3$ analytic expressions for vector coupling coefficients are not known.

Of course one cannot expect that a complicated invariant contraction of SU($N$) vector coupling coefficients can be replaced by an elementary integral, but it is certainly possible to replace it by several or many such integrals. We can only hope that up to a certain order the number of terms generated is small enough to be listed up by a computer. We can say that our result combines the algebra of the group with the combinatorics of the lattice and admits a unified graphical approach to strong coupling expansions.

The method of integrating over the group SU($N$) can be extended to groups U($N$) as well. It can then be compared with the technique developed in [5] which does not yield vector coupling coefficients of U($N$) either. Our method sums up contributions to one irreducible representation which leads to a considerable reduction of the number of terms.

We introduce Bargmann spaces for SU(2) and SU(3) in Sect. 2. Whereas each representation of SU(2) is self-conjugate, representations of SU(3) occur in conjugate pairs in general. Some relevant properties of the conjugation matrix are derived in Sect. 3. Using the delta function kernels for the Bargmann spaces, we integrate tensor products of representation operators $T^R_u$ over the group SU($N$) in Sect. 3. As a final tool we derive the projection kernels that allow us to contract the operators $T^R_u$ to characters $\chi^R(u_{\partial p})$ in Sect. 5. Section 6 is devoted to some miscellaneous remarks on strong coupling expansions and their generating functions.

Notations are the same as in [6], those for links and plaquettes of the lattice