On Estimates of Approximation Numbers and Best Bilinear Approximation

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Abstract. We obtain estimates of approximation numbers of integral operators, with the kernels belonging to Sobolev classes or classes of functions with bounded mixed derivatives. Along with the estimates of approximation numbers, we also obtain estimates of best bilinear approximation of such kernels.

Below we obtain certain estimates of approximation numbers of integral operators, with the kernels belonging to various classes. Let $f(x, y)$ be a continuous $2\pi$-periodic in each variable function. Define the integral operator

$$ (J_f \varphi)(y) = \frac{1}{2\pi} \int_0^{2\pi} f(x, y) \varphi(x) \, dx. $$

It is easy to see that the norm $\|J_f\|_{p \to \infty}$ of the operator $J_f$, acting from $L_p$, $1 \leq p \leq \infty$, into $L_\infty$, equals

$$ \|f(\cdot, y)\|_{p', \infty}, \quad p' = \frac{p}{p - 1}. $$

Denote

$$ \|f(\cdot, y)\|_{p', \infty} = \|f\|_{p', p_2}. $$

Thus

$$ \|J_f\|_{p \to \infty} = \|f\|_{p', \infty}. \tag{1} $$

Note that to define the operators $J_f$ acting from $L_p$ into $L_\infty$, it suffices to assume finiteness of the norm $\|f\|_{p', \infty}$. We define approximation numbers $a_M(J_f)_{p \to q}$, $M = 1, 2, \ldots$, of the operator $J_f$, as the values of best approximation of $J_f$ in operator norm from $L_p$ into $L_q$ by means of a finite-dimensional operator of rank $M$. It is easy to see that (Lemma 3.1)

$$ a_M(J_f)_{p \to \infty} = \inf_{u_i, v_i} \|f(x, y) - \sum_{i=1}^M u_i(x)v_i(y)\|_{p', \infty}. \tag{2} $$


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Given \( f \in L_{p_1, p_2} \), we shall call the quantities
\[
\tau_M(f)_{p_1, p_2} = \inf_{u_i, v_i} \| f(x, y) - \sum_{i=1}^{M} u_i(x)v_i(y) \|_{p_1, p_2}
\]
the best bilinear approximation of the function \( f(x, y) \) in \( L_{p_1, p_2} \). Along with the estimates of approximation numbers, we also obtain here estimates of best bilinear approximation.

We consider the following classes of \( 2\pi \)-periodic functions:

(I) Sobolev classes \( SW^r_B \) of \( r \) times continuously differentiable functions such that
\[
\| f \|_{SW^r_B} = \sum_{0 \leq \beta_1 + \beta_2 \leq r} \left\| \frac{\partial^\beta_1 + \mu_2 f}{\partial x^{\beta_1} \partial y^{\beta_2}} \right\|_q \leq B,
\]
where \( r \) is a positive integer.

(II) Classes \( W^r_B \) of \( 2r \) times continuously differentiable functions such that
\[
\| f \|_{W^r_B} = \sum_{0 \leq \beta_1 \leq r} \left\| \frac{\partial^\beta_1 f}{\partial x^{\beta_1}} \right\|_q \leq B.
\]

If \( B = 1 \), we drop \( B \) in the notations of the classes. We use \( C \) or \( C_l \), \( l = 1, \ldots \), to denote positive absolute constants, whose values may vary.

1. Best Bilinear Approximation

Let \( F \) be a class of functions, \( F \subset L_{p_1, p_2} \). Denote
\[
\tau_M(F)_{p_1, p_2} = \sup_{f \in F} \tau_M(f)_{p_1, p_2}.
\]
Orders of magnitude, with respect to \( M \), of the quantities \( \tau_M(F)_{p_1, p_2} \), for various functional classes \( F \) were computed in [6]-[12]. We prove here the following assertion:

**Theorem 1.1.** The following estimates are valid
\[
\tau_M(SW^r_{\infty})_{1, 1} \geq C_r(M \log M)^{-r},
\]
\[
\tau_M(W^r_{\infty})_{1, 1} \geq C_r M^{-2r}(\log M)^{-r},
\]
where \( C_r \) depends only on \( r \).

**Remark 1.** The known results on the approximation of functions of those classes, by functions of the type
\[
(1.1) \quad \sum_{|k| \leq n} a_k(x)e^{ikx}, \quad \sum_{|k| \leq n} a_k(x)e^{iky} + \sum_{|k| \leq n} b_k(y)e^{ikx},
\]
imply the validity of the upper estimates which are the same as in Theorem 1.1, except for the logarithmic multipliers.