A Generalized Fluctuation-Dissipation Theorem for the One-Dimensional Diffusion Process

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Abstract. The \([x, \beta, \gamma]\)-Langevin equation describes the time evolution of a real stationary process with \(T\)-positivity (reflection positivity) originating in the axiomatic quantum field theory. For this \([x, \beta, \gamma]\)-Langevin equation a generalized fluctuation-dissipation theorem is proved. We shall obtain, as its application, a generalized fluctuation-dissipation theorem for the one-dimensional non-linear diffusion process, which presents one solution of Ryogo Kubo’s problem in physics.

1. Introduction

In order to clarify a probabilistic meaning of the concept of \(T\)-positivity (reflection positivity) with its origin in axiomatic quantum field theory [3, 14], we have investigated a real stationary Gaussian process \(X\) having \(T\)-positivity from the viewpoint of the theory of stochastic differential equations [10, 11, 13]. In the previous paper [11], we characterized a class of stochastic differential equations describing the time evolution of \(X\) as a \([x, \beta, \gamma]\)-Langevin equation and then obtained a fluctuation-dissipation theorem for this \([x, \beta, \gamma]\)-Langevin equation as a generalized fluctuation-dissipation theorem in the theory of Ornstein-Uhlenbeck Brownian motion in statistical physics [2, 6–8, 15].

The purpose of the present paper is to refine the results of [11] and then make them serve to get a generalized second fluctuation-dissipation theorem for the one-dimensional non-linear diffusion process, which presents one solution of Kubo’s problem in physics [6–8]. Before reformulating Kubo’s problem stated in [7], we shall recall briefly a second fluctuation-dissipation theorem for Ornstein-Uhlenbeck Brownian motion. Let \(\mathcal{X}(t), P_x, x \in \mathbb{R}\) be an Ornstein-Uhlenbeck Brownian motion whose time evolution is governed by the following stochastic differential equation:

\[
\begin{align*}
\frac{d\mathcal{X}(t)}{dt} &= -\beta \mathcal{X}(t) + \alpha dB(t) \\
\mathcal{X}(0) &= x.
\end{align*}
\]

(1.1)

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Here \(\alpha\) and \(\beta\) are positive numbers and \((B(t); t \in [0, \infty))\) is a one-dimensional standard Brownian motion. We know that
\[
\lim_{t \to \infty} P_X(\mathcal{X}(t) \in dy) = (\pi \sigma^2 \beta^{-1})^{-1/2} \exp(-y^2/\sigma^2 \beta^{-1}) dy = m_\infty(dy),
\]
and the probability measure \(m_\infty\) is a unique invariant measure for the process \(\mathcal{X}\).

Let \(N = (N(t); t \in \mathbb{R})\) be a stationary Markov process representing the stationary state of the process \(\mathcal{X}\) with \(m_\infty\) as its initial distribution. Then it follows that the time evolution of the process \(N\) is governed by the following stochastic differential equation:
\[
dN(t) = -\beta N(t) dt + \sigma dW(t) \quad (t \in \mathbb{R}).
\]
Here \((W(t); t \in \mathbb{R})\) is a one-dimensional standard Brownian motion. Denoting by \(R\) the covariance function of the process \(N\), we have
\[
R(t) = \frac{\alpha^2}{2\beta} \exp(-\beta |t|) \quad (t \in \mathbb{R}).
\]
From (1.4), we immediately obtain
\[
\frac{\alpha^2}{2} = R(0)\beta.
\]

For the purpose of understanding the physical meaning of formula (1.5), we shall consider the motion of a Brownian particle moving with velocity \(N(t)\) at time \(t\) in a viscous fluid with friction coefficient \(\beta\) whose equation of motion is described by the stochastic differential Eq. (1.3). In that case we can regard the left-hand side in (1.5) as the power of random force causing the zigzag motion of a Brownian particle. On the other hand, (1.2) and (1.5) make us notice that the constant \(R(0)\) is a variance of the equilibrium measure \(m_\infty\), and so we can regard \(R(0) = kT\), where \(k\) is a Boltzman constant and \(T\) is an absolute temperature in the system under consideration. Therefore formula (1.5) stands for a relation between the power of a random force and the friction coefficient of a viscous fluid. And it is to be called a second fluctuation-dissipation theorem. The theoretical ground for a physical understanding of formula (1.5) lies in the stochastic differential Eq. (1.3). That is, it is important, not only from the viewpoint of statistical mechanics, but also from that of probability, to derive a stochastic differential equation describing the time evolution of a stationary process only by using its qualitative nature. The key is how to extract a random force.

Now we shall reformulate Kubo’s problem stated in [7] as follows. Let \(\mathcal{X} = (\mathcal{X}(t), P_x; t \in [0, \infty), x \in \mathbb{R})\) be a one-dimensional diffusion process whose time evolution is governed by the following stochastic differential equation:
\[
\begin{align*}
\mathcal{X}(t) = & b(\mathcal{X}(t)) dt + \sigma(\mathcal{X}(t)) dB(t) \quad (t \in (0, \infty)) \quad \\
\mathcal{X}(0) = & x.
\end{align*}
\]
Here \((B(t); t \in [0, \infty))\) is a one-dimensional standard Brownian motion, \(b\) and \(\sigma\) are continuous functions on \(\mathbb{R}\). Then we know that the Fokker-Planck equation