THE INTEGRAL MODULUS OF CONTINUITY OF THE DIRICHLET KERNEL AND THE CONJUGATE DIRICHLET KERNEL

S. A. Telyakovskii

Asymptotic estimates for the integral modulus of continuity of order s of the Dirichlet kernel and the conjugate Dirichlet kernel are obtained. For example, if \( k\delta \leq \pi/2 \), then

\[
\omega_s(D_k, \delta) = \frac{2^{s+1}}{\pi^2} \sin^s \frac{k\delta}{2} \log \left( 1 + \frac{k}{s} \right) + O \left( 2^s \sin^s \frac{k\delta}{2} \right)
\]

holds uniformly with respect to all the parameters.

Let

\[
\Delta_h^s f(x) := \sum_{m=0}^{s} (-1)^m \binom{s}{m} f(x + mh)
\]

be a difference of function \( f \in L[-\pi, \pi] \) of order \( s \) with step \( h \) at point \( x \). The \( L \)-norm of the difference of function \( f \) will be denoted by

\[
w_s(f, h) := \int_{-\pi}^{\pi} |\Delta_h^s f(x)| \, dx = \int_{-\pi}^{\pi} \left| \Delta_h^s f(x - \frac{sh}{2}) \right| \, dx,
\]

and the integral modulus of continuity of \( f \) by

\[
\omega_s(f, \delta) := \frac{1}{2\pi} \sup_{|h| \leq \delta} w_s(f, h).
\]

To study the integral modulus of continuity of functions with a given sequence of Fourier coefficients, we use estimates of the quantities \( w_s \) and \( \omega_s \) for the Dirichlet kernel and the conjugate Dirichlet kernel

\[
D_k(x) = \frac{1}{2} + \sum_{i=1}^{k} \cos ix, \quad \tilde{D}_k(x) = \sum_{i=1}^{k} \sin ix.
\]

Thus, S. Aljančić [1, Lemma 3] proved that if \( h > 0 \) and \( kh \leq 1 \), then

\[
w_s(D_k, h) \leq c_s h^s k^s \log(k + 1)
\]

and

\[
w_s(\tilde{D}_k, h) \leq c_s h^s k^s \log(k + 1),
\]

where \( c_s \) depends only on \( s \). The author [2, Lemma] showed that the factor \( c_s \) in (4) can be replaced by an absolute constant.

The purpose of this paper is to sharpen estimates (3) and (4). The results obtained give asymptotic estimates for the integral modulus of continuity of the kernels \( D_k \) and \( \tilde{D}_k \).

Let \( c \) denote absolute positive constants which may be different at each occurrence. All the estimates in \( O \)-terms are uniform with respect to all the parameters.

Theorem 1. Let $h > 0$ and $kh \leq \pi$. Then the following estimates hold uniformly with respect to all the parameters:

$$w_s(D_k, h) = \frac{2^{s+2}}{\pi} A_s(k, h) + O \left( 2^s \sin^s \frac{kh}{2} \right),$$

$$w_s(\tilde{D}_k, h) = \frac{2^{s+2}}{\pi} A_s(k, h) + O \left( 2^s \sin^s \frac{kh}{2} \right),$$

where

$$A_s(k, h) = \begin{cases} 
\sin^s \frac{kh}{2} \log \left( 1 + \frac{k}{s \cos \frac{kh}{s}} \right), & \text{if } \cos \frac{kh}{2} > \frac{1}{\sqrt{s}}, \\
\sin^s \frac{kh}{2} \log \left( 1 + \frac{k}{s} \right), & \text{if } \cos \frac{kh}{2} \leq \frac{1}{\sqrt{s}}.
\end{cases}$$

Note that if we reduce the domain of the parameters by replacing the condition $kh \leq \pi$ with $kh \leq \frac{\pi}{2}$, then $\cos \frac{kh}{2} \geq \frac{1}{\sqrt{2}}$ and consequently in this case for all $s$

$$A_s(k, h) = \sin^s \frac{kh}{2} \log \left( 1 + \frac{k}{s} \right) + O \left( \sin^s \frac{kh}{2} \right).$$

It is clear that Theorem 1 contains estimates of type (3) and (4) and sharpens the results formulated above.

The following estimates for integral modulus of continuity are evident corollaries of Theorem 1.

Theorem 2. Let $\delta > 0$ and $k\delta \leq \pi$. Then the estimate

$$\omega_s(D_k, \delta) = \frac{2^{s+1}}{\pi^2} A_s(k, \delta) + O \left( 2^s \sin^s \frac{k\delta}{2} \right)$$

holds uniformly with respect to all the parameters. In particular, if $k\delta \leq \pi/2$, then

$$\omega_s(D_k, \delta) = \frac{2^{s+1}}{\pi^2} \sin^s \frac{k\delta}{2} \log \left( 1 + \frac{k}{s} \right) + O \left( 2^s \sin^s \frac{k\delta}{2} \right).$$

The same estimates are true also for $\omega_s(\tilde{D}_k, \delta)$.

Proof of Theorem 1. We set $\gamma_i := \sin^s \frac{ih}{2}$, $i = 0, \ldots, k$. Since

$$\Delta_h^k \cos i \left( x - \frac{sh}{2} \right) = 2^s \gamma_i \cos \left( ix - \frac{s\pi}{2} \right),$$

$$\Delta_h^k \sin i \left( x - \frac{sh}{2} \right) = 2^s \gamma_i \sin \left( ix - \frac{s\pi}{2} \right),$$

it follows that

$$\Delta_h^k D_k \left( x - \frac{sh}{2} \right) = 2^s \sum_{i=1}^{k} \gamma_i \cos \left( ix - \frac{s\pi}{2} \right),$$

$$\Delta_h^k \tilde{D}_k \left( x - \frac{sh}{2} \right) = 2^s \sum_{i=1}^{k} \gamma_i \sin \left( ix - \frac{s\pi}{2} \right).$$

Therefore, the quantities $w_s(D_k, h)$ and $w_s(\tilde{D}_k, h)$ are equal either to

$$2^s \int_{-\pi}^{\pi} \left| \sum_{i=1}^{k} \gamma_i \cos ix \right| \, dx,$$