OPERATOR DIFFERENTIABLE FUNCTIONS

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A scalar function $f$ is called operator differentiable if its extension via spectral theory to the self-adjoint members of $\mathfrak{B}(H)$ is differentiable. The study of differentiation and perturbation of such operator functions leads to the theory of mappings defined by the double operator integral

$$x \mapsto \int \int \frac{f(\lambda) - f(\mu)}{\lambda - \mu} F(d\mu) x E(d\lambda).$$

We give a new condition under which this mapping is bounded on $\mathfrak{B}(H)$. We also present a means of extending $f$ to a function on all of $\mathfrak{B}(H)$ and determine corresponding perturbation and differentiation formulas. A connection with the “joint Peirce decomposition” from the theory of $JB^*$-triples is found. As an application we broaden the class of functions known to preserve the domain of the generator of a strongly continuous one-parameter group of $*$-automorphisms of a $C^*$-algebra.

1 INTRODUCTION

Let $A$ be a $C^*$-algebra and let $f : \mathbb{R} \to \mathbb{C}$ be a continuous function. By the usual functional calculus $f$ can be extended to a function, also called $f$, on the self-adjoint members of $A$. It is natural to ask when the extended function $f$ is differentiable (we shall always mean Gâteaux differentiable; however, all of the results of which we are aware establish Fréchet differentiability). This problem has been posed by H. Widom [18] in the book “Linear and Complex Analysis Problem Book.” It is not difficult to see that a necessary condition is that the scalar function $f$ must be continuously differentiable. In

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case $A$ is commutative, it is easily seen that this condition is also sufficient. However, in [9] Farforovskaya constructed an example of a $C^1$ function whose extension to the self-adjoint operators on a Hilbert space is not differentiable.

Daletskii and Krein [7] considered this problem in the context of the self-adjoint operators on a Hilbert space $H$. They showed that if $f$ is $C^2$ then its extension is differentiable. Moreover, they obtained a formula for the derivative of $f$ in terms of a notion of "iterated operator integrals", which they also introduced. Birman and Solomyak [2,3,4] refined this concept and introduced and developed the theory of "double operator integrals" which has served as the basis for all subsequent research in this area (see sections 2 and 3). They also found much sharper sufficient conditions for operator differentiability which include, for example, scalar functions whose derivative is Hölder continuous of any order and scalar functions whose derivative has absolutely convergent Fourier series. Still sharper sufficient and necessary conditions were found by Peller [14] (see section 4).

In section 2 and in the first part of section 3 we shall present those parts of this theory upon which the present paper depends. We feel it desirable to include such an exposition since the relevant articles either are in Russian or do not enjoy a wide circulation in English.

Generally speaking, these authors obtained their results upon careful analysis of the Fourier expansion of the scalar function $f$. In section 4 we shall instead proceed by considering a decomposition into Möbius functions. The sufficient condition so obtained is at least as sharp as that of Peller (we don’t yet know if it is strictly sharper). This approach is also more natural and elementary than that via Fourier analysis.

In section 3 we describe a little known functional calculus which allows the scalar function $f$ to be extended to a function which is defined for all bounded operators on $H$. We thus expand the scope of the operator differentiability problem to include such extensions, and provide a sufficient condition for differentiability of $f$ in this case. Along the way we obtain a perturbation formula for functions of bounded operators which is more general than any others of which we are aware.

The functional calculus just referred to is the usual functional calculus for $\mathfrak{B}(H)$