Optimization of plastic conical shells of piece-wise constant thickness

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Abstract This-walled conical shells subjected to uniformly distributed external pressure loading are considered assuming that the thickness of the shells is piece-wise constant. Minimum weight designs are established under the condition that the load carrying capacity of the optimized shell coincides with that of the reference shell of constant thickness. The material of the shell is assumed to obey Tresca's yield condition and the associated flow law. The exact yield surface in the space of generalized stresses corresponding to the Tresca condition is approximated with the squares on the planes of membrane forces and moments, respectively.

1 Introduction
Limit analysis of thin-walled rigid-plastic plates and shells has been studied by Hodge (1960). Ideal rigid-plastic conical shells loaded through a central rigid boss were investigated by Hodge (1960), Onat (1960), Lance and Onat (1963), Lance and Lee (1969), and Hodge and Deruntz (1964). The collapse loads for conical shells subjected to uniformly distributed lateral pressures have been determined for a material obeying Tresca's yield condition by Kuech and Lee (1965). Shells subjected to lateral pressure and edge tension were considered by Hodge and Lakshmikantham (1963).

In a previous study (Lellep and Puman 1994), the authors attempted to minimize the weight of the shell of piece-wise constant thickness. They assumed that the shell structure was loaded at its vertex through a central absolutely rigid boss. In the present paper a minimum weight technique is suggested for conical shells subjected to uniformly distributed lateral pressure. It is assumed that the thickness of the shell is piece-wise constant, whereas the material is rigid-plastic obeying Tresca's yield condition.

2 Formulation of the problem
Let us consider a thin-walled conical shell subjected the uniformly distributed lateral pressure of intensity \( P \). Assume that the shell is simply supported or clamped at the outer edge and absolutely free at the inner edge of radius \( a \) (Fig. 1).

Fig. 1. Shell geometry

The thickness of the shell wall is assumed to be piece-wise constant, e.g.

\[
\begin{align*}
\text{\( h \) = } & \begin{cases} 
\text{\( h_0 \),} & a \leq r < b, \\
\text{\( h_1 \),} & b \leq r \leq R,
\end{cases} \\
\text{\( \text{(1)} \)}
\end{align*}
\]

where \( b, h_0 \) and \( h_1 \) are previously unknown constant parameters.

We are looking for the design of the shell which the weight (material volume) attains the minimum value for a given load carrying capacity. Evidently, the volume of the shell material may be presented as

\[
V_0 = \frac{\pi}{\cos \varphi} \left[ h_0 (b^2 - a^2) + h_1 (R^2 - b^2) \right],
\]

where \( a, R \) and \( \varphi \) are considered as given constants.

In order to establish the minimum weight design of the shell one must minimize (2) so that the equilibrium equations, the associated flow law with geometrical relations and plasticity conditions, are satisfied at each point of the shell.

3 Governing equations
Due to symmetry the stress state of the shell is defined by membrane forces \( N_1, N_2 \) and moments \( M_1, M_2 \). Equilibrium equations of a shell element may be presented as (Hodge 1963; Kuech et al. 1965)

\[
\begin{align*}
\frac{d}{dr} (r N_1) - N_2 &= 0, \\
\frac{d}{dr} \left[ \frac{d}{dr} (r M_1) - M_2 \right] + N_2 \frac{\sin \varphi}{\cos^2 \varphi} + \frac{Pr}{\cos^2 \varphi} &= 0.
\end{align*}
\]

\[
\text{\( \text{(3)} \)}
\]
The corresponding strain rate components are (Kuech et al. 1965)

$$\dot{\varepsilon}_1 = \frac{d\dot{U}}{dr} \cos \varphi, \quad \dot{\varepsilon}_2 = \frac{1}{r} (\dot{U} \cos \varphi - \dot{W} \sin \varphi),$$

$$\dot{\kappa}_1 = -\frac{M_0}{N_0} \frac{d^2 \dot{W}}{dr^2} \cos^2 \varphi, \quad \dot{\kappa}_2 = -\frac{M_0}{N_0} \frac{1}{r} \frac{d\dot{W}}{dr} \cos^2 \varphi.$$  (4)

In (4) $\dot{U}$ and $\dot{W}$ stand for the displacement rates in the normal and circumferential directions, respectively, whereas $M_0 = \sigma_0 h^2/4$, $N_0 = \sigma_0 h$, $\sigma_0$ being the yield stress of the material.

It appears to be reasonable to introduce the following nondimensional quantities:

$$\varrho = \frac{r}{R}, \quad \alpha = \frac{a}{R}, \quad \beta = \frac{b}{R}, \quad \delta = \frac{h_0}{h_*}, \quad \gamma = \frac{h_1}{h_*},$$

$$v = \frac{h}{h_*}, \quad w = \frac{W}{R}, \quad u = \frac{U}{R}, \quad n_{1,2} = \frac{N_{1,2}}{N_*}.$$  (5)

In (5), $M_*$ and $N_*$ stand for the limit moment and limit load for the reference shell of thickness $h_*$. Thus $M_* = \sigma_0 h_*^2/4$, $N_* = \sigma_0 h_*$.

In variables (5) the equilibrium equations (3) take the form

$$(\varrho m_1)' - m_2 = 0, \quad k [(\varrho m_1)']' - m_2 + p \varrho = 0,$$  (6)

where the primes denote differentiation with respect to $\varrho$.

Making use of (5) the strain rate components (4) may be presented as

$$\dot{\varepsilon}_1 = \dot{u}' \cos \varphi, \quad \dot{\varepsilon}_2 = \frac{1}{\varrho} (\dot{u} \cos \varphi - \dot{w} \sin \varphi),$$

$$\dot{\kappa}_1 = -k_0 \dot{w}'' \sin \varphi, \quad \dot{\kappa}_2 = -\frac{k_0}{\varrho} \dot{w}'' \sin \varphi,$$  (7)

where

$$k_0 = \frac{M_0 \cos^2 \varphi}{RN_* \sin \varphi}.$$  (8)

The material of shells is assumed to be rigid-plastic obeying Tresca's yield condition and associated flow (gradientality) law. Tresca's yield condition in its original form is presented as a hexagon on the plane of principal stresses. The yield surface in the space of generalized stresses (membrane forces and moments) is of quite intricate structure. A number of authors tried to replace the exact yield surface with a simpler one so that the load carrying capacities obtained on the basis of the approximated yield surface and the exact one, respectively, do not differ significantly from each other (Hodge 1963; Jones and Ich 1972; Sankaranarayanan 1964).

Hodge (1963) suggested a so-called two-moment limited interaction yield surface which might be presented as two similar hexagons on the planes of bending moments and membrane forces, respectively. Later Sankaranarayanan (1964), and Jones and Ich (1972) suggested further simplifications of the yield surface for rotationally symmetric shells.

In the present study the generalized square yield condition (Fig. 2) will be used. It is assumed that the vectors $\kappa = (\kappa_1, \kappa_2)$ and $\varepsilon = (\varepsilon_1, \varepsilon_2)$ are normal to the squares on planes of bending moments and membrane forces, respectively.

Boundary conditions for stress components are given below. Since the inner edge is free one has

$$n_1(\alpha) = 0, \quad m_1(\alpha) = 0, \quad s(\alpha) = 0,$$  (9)

where $s$ stands for the nondimensional shear force. Evidently, $s = (m_1 \varrho)' - m_2$. At the outer edge

$$m_1(1) = 0, \quad (10)$$

in the case of a simply supported shell and

$$m_1(1) = -\gamma^2,$$  (11)

if the edge is clamped. In the both cases

$$\dot{w}(1) = \dot{w}(1) = 0.$$  (12)

4 Shells of constant thickness

Consider a reference shell of constant thickness $h_*$. It appears that the stress regime corresponds to the sides $C_1 D_1$ and $A_2 B_2$ of the squares presented in Fig. 2 (here $v \equiv 1$). Thus

$$n_2 = -1, \quad m_2 = 1 \text{ over the shell.}$$

Substituting $n_2$ and $m_2$ in (6) and integrating these equations with respect to $\varrho$, making use of boundary conditions one obtains

$$n_1 = \frac{1}{\varrho} (\alpha - \varrho),$$

$$m_1 = \frac{\alpha - \varrho}{k_0} \left[ k + \frac{1}{2} (\varrho - \alpha) - \frac{P}{6} (\varrho - \alpha)(\varrho + 2\alpha) \right].$$  (13)

It immediately follows from (10) and (13) that the load carrying capacity for the simply supported shell is

$$p = \frac{3(2k - \alpha + 1)}{1 + \alpha - 2\alpha^2}. \quad (14)$$

Similarly, the boundary requirement (11) with (13) gives

$$p = \frac{-6\alpha(\alpha - 2) + 3(\alpha - 1)^2}{(1 - \alpha)^2(1 + 2\alpha)},$$  (15)

for a shell with a clamped edge.