One Dimensional $1/|j - i|^s$ Percolation Models: The Existence of a Transition for $s \leq 2$

C. M. Newman$^1\ast$ and L. S. Schulman$^{1,2}$

1 Department of Mathematics and Program in Applied Mathematics, University of Arizona, Tucson, Arizona, 85721, USA
2 Department of Physics, Technion, Haifa, Israel and Department of Physics, Clarkson University, Potsdam, New York, 13676, USA

Abstract. Consider a one-dimensional independent bond percolation model with $p_j$ denoting the probability of an occupied bond between integer sites $i$ and $i \pm j, j \geq 1$. If $p_j$ is fixed for $j \geq 2$ and $\lim_{j \to \infty} j^2 p_j > 1$, then (unoriented) percolation occurs for $p_1$ sufficiently close to 1. This result, analogous to the existence of spontaneous magnetization in long range one-dimensional Ising models, is proved by an inductive series of bounds based on a renormalization group approach using blocks of variable size. Oriented percolation is shown to occur for $p_1$ close to 1 if $\lim_{j \to \infty} j^s p_j > 0$ for some $s < 2$. Analogous results are valid for one-dimensional site-bond percolation models.

1. Introduction and Main Results

We consider translation-invariant one-dimensional independent site-bond percolation models in which each site $i \in \mathbb{Z}$ is alive (respectively dead) with probability $\lambda$ (respectively $1 - \lambda$) and in which the (non-directed) bond between any distinct $i, j \in \mathbb{Z}$ is occupied (respectively vacant) with probability $p_{|i-j|}$ (respectively $1 - p_{|i-j|}$). All the sites and bonds are mutually independent. We will treat both nonoriented and oriented percolation. In ether case the cluster of $i$, $C(i)$, consists of those living sites for which there is a path of occupied bonds starting at $i$, ending at $j$, and touching only living sites; in particular $i \in C(i)$ if and only if $i$ is alive. In nonoriented percolation, any such path is allowed; in oriented percolation only paths that move to the right at each step are allowed. Such site-bond models reduce to pure bond models when $\lambda = 1$ and to pure site models when each $p_j = 0$ or 1.

A special case is bond percolation with $\lambda = 1$ and $p_j = 1 - \exp(-\beta |j|^{-s})$ for some $s, \beta \geq 0$. It is an elementary fact that for $s \leq 1$, percolation occurs (i.e., $P_\infty \equiv P(\|C(0)\| = \infty) > 0$, where $\|C\|$ denotes the number of sites in $C$) for any $\beta > 0$;
the fact that moreover (in the nonoriented case) \( C(0) = \mathbb{Z} \) [GKM] will not concern us in this paper. It can also be shown that for \( s > 2 \), percolation does not occur for any \( \beta \), while for \( 1 < s \leq 2 \), percolation does not occur for small \( \beta \) (i.e., when \( 2 \sum_{j=1}^{\infty} p_j < 1 \) [Sc]. Our first main result is a proof that for \( 1 < s < 2 \), oriented (and a fortiori nonoriented) percolation (and hence a phase transition) occurs for sufficiently large \( \beta \). This is analogous to the occurrence of a phase transition in long-range one-dimensional Ising models having an \( |i-j|^{-s} \) interaction with \( 1 < s < 2 \) [D]. Our second main result (which applies only in the nonoriented case) is that for \( s = 2 \), percolation (and hence a phase transition) occurs for large \( \beta \). This is analogous to the corresponding Ising model result [FS].

We note that the question of percolation in long-range one-dimensional models was posed by Erdős several years ago [Sh]. In [AN], further results for the \( s = 2 \) case are presented which go beyond what was currently known for Ising models. In [ACCN], the results of [AN] are extended to Ising (and Potts) models; in the process, the relation between the percolation results presented here and the Ising results of [FS] is clarified.

We conclude this section by stating our main results for general one-dimensional site-bond models whose \( p_j \)'s satisfy appropriate asymptotic hypotheses. Theorem 1.1 corresponds to oriented percolation for \( 1 < s < 2 \) and Theorem 1.2 to unoriented percolation for \( s = 2 \). Our results on the occurrence of unoriented percolation are, in a certain sense, optimal (see Remarks 1.3 and 2.4 below). Our results on the occurrence of oriented percolation can perhaps be improved. It is not clear, for example, whether oriented percolation can occur for \( s = 2 \). The proofs of Theorems 1.1 and 1.2 are based on a renormalization group analysis. In Sect. 2, this analysis is introduced in terms of natural “continuum-bond” models with simple scaling properties. It is then shown that the proofs of our main results can be reduced to certain asymptotic results concerning iterated mappings on two (for Theorem 1.1) or three (for Theorem 1.2) dimensional parameter spaces. In Sect. 3 (for Theorem 1.1) and Sect. 4 (for Theorem 1.2) these iterated mapping results are shown to be valid. As expected, there are many more technical details involved in the proof of Theorem 1.2 since it treats the critical value of \( s \). It may be worth remarking that although most renormalization group analyses are approximate or heuristic, ours produces rigorous bounds. See [ACCFR] for another (but different) rigorous renormalization group argument in a percolation context. See [AYH] for a pioneering (nonrigorous) renormalization group analysis of \( 1/|j - i|^2 \) Ising models.

In the statements of both theorems, \( p_1 \) and \( \lambda \) are regarded as parameters with \( p_j \) fixed for \( j > 1 \). \( \lambda \) will be one of the parameters in both parameter spaces introduced in Sect. 2. There will also be a parameter \( \beta \), essentially \( \lim_{j \to \infty} j^s p_j \), which controls the long range behavior of the model. In the two parameter space (for Theorem 1.1) the short range parameter \( p_1 \) is dropped while in the three parameter space (for Theorem 1.2) it is replaced by a closely related cutoff parameter \( \xi \).

**Theorem 1.1.** If a one-dimensional site-bond model has \( \liminf_{j \to \infty} j^s p_j > 0 \) for some \( s < 2 \), then (with \( p_j \) fixed for \( j > 1 \)) percolation (both oriented and nonoriented) occurs when \( p_1 \) and \( \lambda \) are sufficiently close to 1.