The Completeness of Functional Logic

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Abstract. A new first-order logic, functional logic, was proposed recently by Staples, Robinson and Hazel. The logic provides a formal means of describing and reasoning about dependence on an implicit parameter, a prime motivation being the unification of the Hoare logic used to reason about procedural programs with the powerful and well established techniques of classical logic. Viewed more abstractly, independently of possible applications, functional logic may be described as a logic with primitives, axioms and inference rules appropriate for reasoning about the properties of mathematical functions. In this paper, the completeness of functional logic is proved; that is, it is shown that any term of a theory in the logic which is true in all models is a theorem.

1. Introduction

This paper presents a completeness theorem for functional logic, a new first order logic introduced by Staples, Robinson and Hazel in [Sta93], with which the present paper appears.

Functional logic was conceived as a language in which the notions and arguments required for high-level reasoning about computation could be satisfactorily formalised. Specifically, the logic allows formal description of, and reasoning about, dependence on an implicit parameter, allowing the integration of

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the Hoare logic [Hoa69, Apt81] used to reason about procedural programs with the powerful and well established techniques of classical logic. In computational contexts, the relevant implicit parameter is frequently a suitable abstraction of program state, but in other theories it might, for example, be time. Accounts of a formal functional set theory grounded in functional logic, and its use in modelling programming concepts, are available [HeS90, NiH92, HEN92, Hen91]. For the purposes of a discussion of completeness, it is useful to view functional logic abstractly, independently of its potential applications. From this viewpoint, the logic may simply be described as one having primitives, axioms and inference rules appropriate for reasoning about the properties of mathematical functions. The standard semantics for functional logic reflects this description: the universe of discourse of any interpretation or model is a suitable set of functions. The completeness theorem proved in this paper shows that any term in a functional theory which is true in all such functional models of the theory must be a theorem.

Since [Sta93] and this paper appear together, we do not reproduce or summarise any of the material of [Sta93] here, and we assume that the reader is familiar with all this material. In particular, the motivations behind the introduction of functional logic, the relation of functional logic to other formalisms, the syntax and semantics of the logic, mechanisms for the introduction of defined symbols, and the conflation of the notions of term and formula, all of which are treated fully in [Sta93], are assumed understood here.

The structure of the paper is as follows. In Section 2 we give the definitions, terminology and notation needed for the remainder of the paper, and in Section 3 we derive a functional version of the Deduction Theorem, which is essential for later arguments. In Sections 4 and 5 we state and prove the completeness theorem. Section 5 is devoted to the key technical step of the argument, the construction of a certain model, while the briefer Section 4 presents the remainder of the completeness proof on the basis of this construction. A number of formal theorems in functional logic are needed at various points, most notably in Section 5, and these are collected in Section 6. In the interests of brevity, following the advice of a referee, proofs of these results are not given here. Full proofs are available in [Nic90].

2. Preliminaries

2.1. Functional Theories, Interpretations and Models

A functional theory is a theory consisting of functional logic, sets (possibly empty) of extra, non-logical function and quantifier symbols, and a set of terms of the theory, which are the theory's proper axioms.

Throughout, we take \( x, y, z, \ldots \) (possibly with digits as suffixes) to be variables belonging to the primitive syntax of functional logic, and we use \( A, B, C, \ldots, S, T, U, \ldots \) (possibly with suffixes) as informal meta-level variables ranging over functional terms.

We follow [Sta93] in saying that an interpretation of a functional theory is defined by an index set \( I \) and a set \( U \) of strict functions on \( I \), the standard functional interpretations of the logical function and quantifier symbols, and arbitrary interpretations of the non-logical symbols as members of \( U \), functions on \( U \) or quantifiers on \( U \), as appropriate. We write \( \mathcal{S} \) for a chosen interpretation of a symbol.