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Inductive Families[1]

Peter Dybjer

Department of Computer Sciences, Chalmers University of Technology, Göteborg, Sweden

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Abstract. A general formulation of inductive and recursive definitions in Martin-Löf's type theory is presented. It extends Backhouse's 'Do-It-Yourself Type Theory' to include inductive definitions of families of sets and definitions of functions by recursion on the way elements of such sets are generated. The formulation is in natural deduction and is intended to be a natural generalisation to type theory of Martin-Löf's theory of iterated inductive definitions in predicate logic.

Formal criteria are given for correct formation and introduction rules of a new set former capturing definition by strictly positive, iterated, generalised induction. Moreover, there is an inversion principle for deriving elimination and equality rules from the formation and introduction rules. Finally, there is an alternative schematic presentation of definition by recursion.

The resulting theory is a flexible and powerful language for programming and constructive mathematics. We hint at the wealth of possible applications by showing several basic examples: predicate logic, generalised induction, and a formalisation of the untyped lambda calculus.

1. Introduction

The inference rules of Martin-Löf's type theory can be separated into three parts:

- general rules;
- rules for ordinary set formers;
- rules for universes.

[1] A shorter preliminary version of this paper appeared under the title 'An inversion principle for Martin-Löf's type theory' in the Proceedings of the Workshop on Programming Logic in Båstad, May 1989, Programming Methodology Group Report 54, Chalmers University of Technology and the University of Göteborg, pages 177-190.

Correspondence and offprint requests to: Peter Dybjer, Department of Computer Sciences, Chalmers University of Technology, S-412 96 Göteborg, Sweden.

Typically the second part includes the set formers Π, Σ, $+$, I, N_n, N, W and *List* (Martin-Löf [MaL84]), but it is often remarked that this collection can be extended when there is a need for it. However, the desire for such extensions is so common that general principles need to be layed down which ensure their correctness. There are two possibilities:

- using a general purpose construction which is part of the theory;
- giving external criteria for correct extensions of the theory.

An example of a general purpose construction is *impredicative quantification*, which is part of system F, the calculus of constructions [CoH88], and related systems.

This construction is not part of Martin-Löf's type theory, which is *predicative*. But in the extensional version [MaL84] *wellorderings* W can be used instead. It can for example be proved [Dyb88] that for any strictly positive set operator Φ built up by constants, variables, $+$, $\times$, and $\rightarrow$, there is a set A and a family of sets B over A, such that

$$\Phi(X) \cong \Sigma_{x:A} X^{B(x)}.$$

Since $W_{x:A}B(x)$ satisfies the isomorphism

$$X \cong \Sigma_{x:A} X^{B(x)}$$

we are justified in using it as a representation for the set generated inductively by Φ. But this method does not work in the intensional version of type theory given by Martin-Löf in 1986 [MaL86, NPS90]), since it makes use of 'extensional isomorphisms', such as $N_1 \cong X^{N_0}$ and $X \cong X^{N_1}$.

Another possibility is to add *fixed point operators* to type theory. But the formulation of Mendler [Men87] has the drawback that it needs a notion of subtype and therefore require fundamental changes of the theory. A new formulation which does not assume a notion of subtype has been proposed by Coquand and Paulin [CoP90].

The second possibility is the topic of this paper: to specify a scheme which determines correct extensions of a theory. Thus we do not deal with a fixed theory but an *open* theory. But note that what we consider fixed and open is a matter of convention. We can make the theory closed by formalising rules for sequents

$$T;\Gamma \vdash \mathscr{J},$$

where a judgement $\mathscr{J}$ is made in a current theory T as well as in a context of assumptions Γ, and where there are formal rules for a correct current theory T as well as for a correct context Γ. (Compare the presentation of the rules for the Edinburgh LF system in Harper, Honsell, and Plotkin [HHP87], which contain rules for correct *signatures* as well as for contexts. Signatures play a similar rule to our current theories, but cannot contain definitional equalities. Moreover, correctness of a signature is only a form of type correctness which does not require that the constants are interpreted in terms of inductive definitions.)

The main point here is that *Martin-Löf's type theory is a theory of inductive definitions formulated in natural deduction.* Each set former (logical constant) is defined inductively by its introduction rules. The elimination rule expresses a principle of definition by recursion (proof by induction). Equality rules express how these definitions are eliminated (proofs are normalised).

First we specify what it means to be a correct definition of a set former by