\textbf{$L^1$ Sensitivity Minimization for Plants with Commensurate Delays*}

Munther A. Dahleh† and Yoshito Ohta‡

Abstract. In this paper we consider the problem of $L^1$ sensitivity minimization for linear plants with commensurate input delays. We describe a procedure for computing the minimum performance, and we characterize optimal solutions. The computations involve solving a one-parameter family of finite-dimensional linear programs. Explicit solutions are presented for important special cases.

Key words. $L^1$ optimal control, Delay and infinite-dimensional linear systems, Linear programming.

\textbf{Notation}

\begin{itemize}
  \item $X^*$ Dual space of a normed linear space $X$.
  \item $\mathcal{B}S$ All elements in $S$ with norm $\leq 1$.
  \item $S^\perp$ The annihilator subspace defined as
    \[ \{ x^* \in X^* | \langle s, x^* \rangle = 0 \text{ for all } s \in S \subset X \} \] 
  \item $S^\perp$ The annihilator subspace defined as
    \[ \{ x \in X | \langle x, s \rangle = 0 \text{ for all } s \in S \subset X^* \} \]
  \item $\BV(X)$ Functions of bounded variation on $X$.
  \item $C_0(X)$ Continuous function on a locally compact space $X$ such that
    \[ \text{for all } \varepsilon > 0, \quad \{ x \in X | |f(x)| \geq \varepsilon \} \text{ is compact} \]
  \item $C^N(a, b)$ Vectors of continuous functions on $(a, b)$.
\end{itemize}

\section{1. Introduction}

The design of controllers for infinite-dimensional linear plants has received considerable attention in the literature. The interest stems from the fact that there are
delays present in every system, and that many systems are governed by partial differential equations. Recently, a lot of effort has been devoted to generalizing the celebrated $H^\infty$ control theory for plants with delays. Results have been reported in various papers, including [FM], [FTZ], and [T].

The $L^1$ design methodology is a recent theory motivated in [V2] and solved in the rational case in [DP]. The solution was obtained using duality theory, and the problem was shown to be equivalent to a finite-dimensional programming problem. A complete solution for the case of an arbitrary rational plant with a pure input delay was furnished in [D] and was shown to be equivalent to a modified rational problem and, hence, can be solved by using the theory in [DP].

In this paper, we consider plants of the form $P = UP_0$ where $P_0$ is stable with a stable inverse, and $U$ is a polynomial of delays. We will provide procedures for computing the minimum performance by solving linear programming problems, and for constructing optimal solutions. The method of solution is quite different from the one presented in [D] for the case of pure delay, which cannot be generalized.

In the $H^\infty$ problem, the solution in the case of infinite-dimensional plants, e.g., plants with delays, is essentially obtained by the computation of an associated Hankel operator, which might not even be a compact operator. In the $L^1$ problem, such a characterization of optimal solutions does not exist, and hence we will extend the dual problem formulation to solve the above problem. It is our hope that this solution will be the first step in solving the general problem of infinite-dimensional systems.

2. Mathematical Preliminaries

In this section, we introduce some of the mathematical preliminaries needed in the sequel. Let $AM$ denote the space of all atomic measures of bounded variation on $\mathbb{R}_+$, i.e.,

$$AM = \left\{ h \mid h(t) = \sum_{i=0}^{\infty} h_i \delta(t - t_i), \{ h_i \} \in l^1, t_i \geq 0 \right\}$$

with norm defined as

$$\| h \|_{AM} = \sum_{i=0}^{\infty} | h_i |.$$

The algebra $A$ [DV] is the direct sum of $AM$ and $L^1(\mathbb{R}_+)$. Hence $A$ consists of all distributions of the form

$$h(t) = h_{AM}(t) + h_M(t),$$

where $h_{AM} \in AM$ and $h_M \in L^1(\mathbb{R}_+)$. The norm on $A$ is defined as

$$\| h \|_A = \| h_{AM} \|_{AM} + \| h_M \|_{L^1(\mathbb{R}_+)}.$$

We will denote by $\hat{A}$ the space of Laplace transforms of distributions in $A$, with the same norm. It is well known that $A$ is a Banach algebra with the standard convolution, and defines a set of BIBO-stable, linear, time-invariant impulse responses with the given operator norm. For the rest of this paper, $H$ will denote the Laplace transform of $h$, i.e., $H \in \hat{A}$ and $h \in A$. 