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Central Charges in the Canonical Realization of Asymptotic Symmetries: An Example from Three Dimensional Gravity

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Abstract. It is shown that the global charges of a gauge theory may yield a nontrivial central extension of the asymptotic symmetry algebra already at the classical level. This is done by studying three dimensional gravity with a negative cosmological constant. The asymptotic symmetry group in that case is either $R \times SO(2)$ or the pseudo-conformal group in two dimensions, depending on the boundary conditions adopted at spatial infinity. In the latter situation, a nontrivial central charge appears in the algebra of the canonical generators, which turns out to be just the Virasoro central charge.

I. Introduction

In general relativity and in other gauge theories formulated on noncompact ("open") spaces, the concept of asymptotic symmetry, or "global symmetry," plays a fundamental role.

The asymptotic symmetries are by definition those gauge transformations which leave the field configurations under consideration asymptotically invariant, and recently, it has been explicitly shown that they are essential for a definition of the total ("global") charges of the theory [1, 2]. (For earlier connections between asymptotic symmetries and conserved quantities in the particular case of Einstein theory, see [3, 4] and references therein.)

The basic link between asymptotic symmetries and global charges has been emphasized again in recent papers dealing with the monopole sector of the SU(5) grand unified theory [5] and with $D = 3$ gravity and supergravity [6], where it is confirmed that the absence of asymptotic symmetries prohibits the definition of global charges. In the first instance, the unbroken symmetry group of the monopole solution is not contained in the set of asymptotic symmetries because of topological obstructions. This forbids the definition of meaningful global color charges

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associated with the unbroken group. In the second case, the nontrivial global properties of the conic geometry, which describes the elementary solution of $D = 3$ gravity, prevents the existence of well defined spatial translations and boosts, and hence, also of meaningful linear momentum and "Lorentz charge."

In the Hamiltonian formalism, the global charges appear as the canonical generators of the asymptotic symmetries of the theory: with each such infinitesimal symmetry ξ is associated a phase space function $H[\xi]$ which generates the corresponding transformation of the canonical variables. It is generally taken for granted that the Poisson bracket algebra of the charges $H[\xi]$ is just isomorphic to the Lie algebra of the infinitesimal asymptotic symmetries, i.e., that

$$\{H[\xi], H[\eta]\} = H[[\xi, \eta]]. \tag{1.1}$$

The purpose of this paper is to analyze this question in detail.

It turns out that, while (1.1) holds in many important examples, it is not true in the generic case. Rather, the global charges only yield a "projective" representation of the asymptotic symmetry group,

$$\{H[\xi], H[\eta]\} = H[[\xi, \eta]] + K[\xi, \eta]. \tag{1.2}$$

In (1.2), the "central charges" $K[\xi, \eta]$, which do not involve the canonical variables, are in general nontrivial, i.e., they cannot be eliminated by the addition of constants C_ξ to the generators $H[\xi]$.

The occurrence of classical central charges is by no means peculiar to general relativity and gauge theories, and naturally arises in Hamiltonian classical mechanics ([7] appendix 5). It results from the non-uniqueness of the canonical generator associated with a given (Hamiltonian) phase space vector field. Indeed, this generator is only determined up to the addition of a constant, which commutes with everything. Accordingly, the Poisson bracket of the generators of two given symmetries can differ by a constant from the generator associated with the Lie bracket of these symmetries.

A similar phenomenon occurs with asymptotic symmetries in gauge theories. In that case, the Hamiltonian generator $H[\xi]$ of a given asymptotic symmetry ξ^A differs from a linear combination of the constraints $\phi_A(x)$ of the canonical formalism by a surface term $J[\xi]$ which is such that $H[\xi]$ has well defined functional derivatives [8],

$$H[\xi] = \int d^n x \xi^A(x) \phi_A(x) + J[\xi]. \tag{1.3}$$

But this requirement fixes $J[\xi]$, and hence $H[\xi]$, only up to the addition of an arbitrary constant. This ambiguity signals the possibility of central charges.

Because the theory of central charges in classical mechanics is well known [7], we will only discuss here the aspects which are peculiar to gauge theories and asymptotic (as opposed to exact) symmetries. This will be done by treating three dimensional Einstein gravity with a negative cosmological constant Λ in detail. In that instance, we show that the asymptotic symmetry group is either $R \times SO(2)$, or the conformal group in two dimensions, depending on the boundary conditions