The Spectrum of a Schrödinger Operator in \( L_p(\mathbb{R}^n) \) is p-Independent

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Abstract. Let \( H_p = -\frac{1}{2}\Delta + V \) denote a Schrödinger operator, acting in \( L_p(\mathbb{R}^n) \), \( 1 \leq p \leq \infty \). We show that \( \sigma(H_p) = \sigma(H_2) \) for all \( p \in [1, \infty] \), for rather general potentials \( V \).

Introduction. In [12, 13], B. Simon conjectured that \( \sigma(H_p) \) is p-independent, where \( H_p = -\frac{1}{2}\Delta + V \) is a general Schrödinger operator in \( L_p(\mathbb{R}^n) \). Partial results on this problem are contained in Simon [12], Sigal [10], Hempel, Voigt [5].

In the notations of Sect. 1, our main result reads as follows.

Theorem. Let \( V = V_+ - V_- \), \( V_+ \geq 0 \), where \( V_+ \) is admissible, and \( V_- \in \mathcal{K}_v \) with \( c_0(V_-) < 1 \). Then \( \sigma(H_p) = \sigma(H_2) \) for \( 1 \leq p \leq \infty \).

In addition, if \( \lambda \) is an isolated eigenvalue of finite algebraic multiplicity \( k \) of \( H_p \), for some \( p \in [1, \infty] \), then the same is true for all \( p \in [1, \infty] \).

The proof of this result is contained in Propositions 2.1, 3.1, and 2.2.

In Sect. 2 we prove the inclusion \( \sigma(H_2) \subset \sigma(H_p) \), following ideas of Simon and Davies.

In Sect. 3 we show that the integral kernel of \( (H_2 - z)^{-n} \), for \( n \in \mathbb{N}, n > v/2 \), defines an analytic \( \mathcal{B}(L_p(\mathbb{R}^n)) \)-valued function on \( \rho(H_2) \), which coincides with \( (H_p - z)^{-n} \) for \( z \) real and sufficiently negative. This implies \( \sigma(H_p) \subset \sigma(H_2) \), by unique continuation.

A different situation, where an integral kernel determines operators with p-dependent spectrum, can be found in Jörgens [6; IV, Aufg. 12.11 (b)]; note that the kernel in Jörgens' example is the resolvent kernel of the differential operator

\[- \frac{d}{dx} x^2 \frac{d}{dx} \quad \text{on} \quad (0, \infty), \quad \text{at} \quad z = -2.\]

1. Schrödinger Operators in \( L_p(\mathbb{R}^n) \)

First we recall briefly several facts concerning the semigroup associated with the heat equation. For brevity, we shall write \( L_p \) instead of \( L_p(\mathbb{R}^n) \), in the sequel
(analogously, \( C_c^\infty := C_c^\infty(\mathbb{R}^n) \), etc.). For \( t \in \mathbb{C} \), \( \text{Re} \ t > 0 \), we define \( k_t \in L_1 \) by

\[
k_t(x) := (2\pi t)^{-n/2} \exp\left(-|x|^2/2t\right).
\]

For \( 1 \leq p \leq \infty \) we define \( U_{0,p}(t) \in \mathcal{B}(L_p) \) (\( t \in \mathbb{C} \), \( \text{Re} \ t > 0 \)) by

\[
U_{0,p}(t)f := k_t * f \quad (f \in L_p),
\]

and further \( U_{0,p}(0) = I \). For \( 1 \leq p < \infty \), \( U_{0,p}(\cdot) \) is a holomorphic semigroup of angle \( \pi/2 \); let \( -H_{0,p} \) denote its generator. Further denote \( H_{0,\infty} := H_{0,1}^* \).

Next we introduce the class of potentials \( V \) to be considered in this paper. Following Voigt [14], we define classes of potentials by

\[
\mathcal{K}_v := \left\{ V \in L_{1,\text{loc}}; \text{ess sup} \int_{|x-y| \leq 1} |g_v(x-y)||V(y)|dy < \infty \right\},
\]

where \( g_v \) is the usual fundamental solution of \( \frac{1}{2} \Delta \). Note that this class is slightly larger than the class \( K_v \) in Aizenman, Simon [1], Simon [13]. For \( V \in \mathcal{K}_v \) we define

\[
c_v(V) := \lim (\text{ess sup} \int_{|x-y| \leq 1} |g_v(x-y)||V(y)|dy).
\]

Obviously \( \mathcal{K}_v \subset L_{1,\text{loc},\text{unif}} \) for all \( v \in \mathbb{N} \), \( \mathcal{K}_1 = L_{1,\text{loc},\text{unif}} \), and \( c_1(V) = 0 \) for all \( V \in \mathcal{K}_1 \).

A potential \( V \geq 0 \) will be called \textit{admissible} if \( Q(H_{0,2}) \cap Q(V) \) is dense in \( L_2 \); cf. Voigt [14]. In particular, \( V \geq 0 \) is admissible if \( V \in L_{1,\text{loc}}(G) \), where \( G = \tilde{G} \subset \mathbb{R}^n \) is such that \( \mathbb{R}^n \setminus G \) is a (closed) set of Lebesgue measure zero.

Throughout this paper we shall assume

\[
V = V_+ - V_-, \quad V_+ \geq 0,
V_- \in \mathcal{K}_v \quad \text{with} \quad c_v(V_-) < 1, \quad V_+ \text{ admissible.} \quad (1.1)
\]

In the following proposition we denote the truncation of \( V \) by

\[
V^{(n)} := (\text{sgn} \ V)(|V| \wedge n) \quad (n \in \mathbb{N}).
\]

1.1. Proposition. Let \( V \) satisfy (1.1), and let \( 1 \leq p < \infty \). Then, for \( t \geq 0 \), the limit

\[
U_p(t) := s - \lim_{n \to \infty} \exp(-t(H_{0,p} + V^{(n)}))
\]

exists, and \( (U_p(t); t \geq 0) \) is a \( C_0 \)-semigroup on \( L_p \). The Feynman-Kac formula

\[
U_p(t)f(x) = E_x \left\{ \exp \left(-\int_0^t V(b(s))ds \right) f(b(t)) \right\}
\]

holds for all \( f \in L_p \).

Here, \( E_x \) and \( b(\cdot) \) are as in Simon [13]; cf. Reed, Simon [9], Simon [11]. The proof of this proposition can be found in Voigt [14; Proposition 5.8(a), Proposition 2.8, Remark 5.2(b), Proposition 3.2, Proposition 6.1(c)].

We denote the generator of \( (U_p(t); t \geq 0) \) by \( -H_p \), for \( 1 \leq p < \infty \), and we shall henceforth write \( U_p(t) = \exp(-tH_p) \). Also, \( H_\infty = H_1^* \). More detailed information about the operators \( H_p \), in particular for \( p = 1, p = 2 \) can be found in Voigt [14].