EVERY ARRANGEMENT EXTENDS TO A SPREAD

JACOB E. GOODMAN*, RICHARD POLLACK†, REPHAEL WENGER‡, and TUDOR ZAMFIRESCU &

Received December 5, 1991
Revised July 15, 1992

An arrangement of pseudolines in the Euclidean plane $E^2$ is a finite family of simple curves, each asymptotic to some line at both “ends”, every two of which intersect at precisely one point, at which they cross. Such arrangements, which exhibit many of the properties of straight line arrangements, have been studied since the work of F. Levi [2]; see [1] for an extensive bibliography up to 1971.

A spread of pseudolines in $E^2$ is the continuous version of an arrangement: an infinite family of simple curves, each asymptotic to some line at both “ends”, such that:

1. every two curves intersect at precisely one point, at which they cross;
2. there is a bijection $L$ from the unit circle $C$ to the family of curves such that $L(p)$ is a continuous function of $p \in C$.

Motivated by the fact that any finite arrangement of straight lines can be extended to a spread of straight lines, B. Grünbaum conjectured in [1] that the same should hold for pseudolines. It is this conjecture that we establish below:

**Theorem.** Every arrangement of pseudolines in $E^2$ may be extended to a spread of pseudolines.

**Proof.** We begin by mapping $E^2$ to the interior of a disk, in such a way that pseudolines in $E^2$ map to curves on the disk with endpoints on the circle bounding the disk. Moreover it is easily seen by induction that an arrangement of pseudolines in a disk is combinatorially equivalent to an arrangement of (piecewise linear)

* Supported in part by NSA grant MDA904–89–H–2038, PSC-CUNY grant 662330, and the Center for Discrete Mathematics and Theoretical Computer Science (DIMACS), a National Science Foundation Science and Technology Center under NSF grant STC88–09648.
† Supported in part by NSF grant CCR–8901484, NSA grant MDA904–89–H–2030, and DIMACS.
‡ Supported in part by DIMACS.
& Supported in part by DIMACS.

AMS subject classification code (1991): 51 H 10, 51 A 45; 05 B 99, 51 D 20, 52 A 10, 52 C 99, 57 N 05
pseudolines in a regular 2n-gon such that each face in the arrangement is a convex polygon, the pseudolines having antipodal vertices on the 2n-gon as endpoints. Hence to prove the theorem we need only show that such a finite family of polygonal pseudolines in the 2n-gon can be extended to an infinite family in which every point $p$ on the boundary lies on exactly one pseudoline $L(p)$, with $L(p)$ a continuous function of $p$.

Let $l$ and $l'$ be two piecewise linear curves in the 2n-gon $P$ with distinct antipodal endpoints $p$, $\bar{p}$ and $p'$, $\bar{p}'$, respectively, and let us drop for a moment the restriction that they meet (and cross) at precisely one point. Let $q$ be an isolated point of intersection of $l$ and $l'$ at which they cross. Thus some small topological disk $\Delta$ contains $q$ and no other point of intersection of $l$ and $l'$, and $l$ and $l'$ intersect $\partial\Delta$ at four points, $s$, $s'$, $\bar{s}$, and $\bar{s}'$, lying between $q$ and $p$, $p'$, $\bar{p}$, and $\bar{p}'$, respectively. (See Figure 1.) We say that $q$ is a proper intersection point of $l$ and $l'$ if $l$ and $l'$ cross at $q$ and if $s$, $s'$, $\bar{s}$, $\bar{s}'$ occur in the same order around $\Delta$ (clockwise or counterclockwise) as $p$, $p'$, $\bar{p}$, $\bar{p}'$ do around $P$.

![Fig. 1. q is a proper intersection point, Q* is not](image)

We can then replace the global condition that curves intersect at precisely one point, at which they cross, by the local condition that every point of intersection is proper.

**Lemma 1.** Two piecewise linear curves with antipodal endpoints on a disk intersect at precisely one point, at which they cross, if and only if every point of intersection of the two curves is a proper intersection point.

**Proof.** Since the endpoint of each curve are antipodal, the first curve separates the endpoints of the second and thus the curves must have at least one intersection point. If there is only one, that intersection must clearly be proper. One the other hand, if two piecewise linear curves $l$ and $l'$ intersect at more than one point, we can list the points of intersection in order along $l$. Let $q$ and $q^*$ be two successive points of intersection. It is not hard to see that if $q$ is proper then $q^*$ is not. ■