Controllability by Completions of Partial Upper Triangular Matrices*

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Abstract. Given an irreducible partial upper triangular \( n \times n \) matrix \( A \), it is shown that for every nonzero vector \( b \) there exists a completion \( A_c \) of \( A \) such that the pair \( (A_c, b) \) is controllable. Various extensions and applications of this result are given.

Key words. Controllability, Completion of matrices, Pole assignment.

1. Introduction

We consider matrices with entries in a fixed field \( F \) (in this paper, either \( F = \mathbb{R} \), the real numbers, or \( F = \mathbb{C} \), the complex numbers). An \( n \times n \) matrix \( A \) is called partial if some entries of \( A \) are given, or specified (as elements in \( F \)), while the other entries are free independent variables that take values in \( F \). If the variable entries in a given partial matrix \( A \) assume specific values, then an \( n \times n \) matrix \( A_c \) with entries in \( F \) results; \( A_c \) is called a completion of \( A \).

Various completion problems of partial matrices have been, and continue to be, extensively studied, largely due to numerous applications. Typically, a completion problem consists of finding, if possible, a completion of a given partial matrix \( A \) with specified properties, or describing all such completions. The specified properties may be positive definiteness, having norm less than 1, having a given set of eigenvalues, etc. It is impossible to provide here a comprehensive list of papers on completion problems; we mention only [BGRS], [GRS], and [RS], which are the most relevant to the subject of this paper.

In this paper we focus on the controllability property. Recall that a pair \( (A, b) \), where \( A \) is an \( n \times n \) matrix and \( b \) is an \( n \)-dimensional vector, is called controllable if \( b, Ab, \ldots, A^{n-1}b \) form a basis of \( F^n \). (Several other equivalent definitions are well known.) As a simple example, let \( A_0 = [a_{ij}]_{i,j=1}^n, \quad b_0 = [b_i]_{i=1}^n \), where \( a_{i,i+1} = 1 \) \( (i = 1, \ldots, n - 1) \), \( a_{ij} = 0 \) if \( j - i \geq 2 \) (Hessenberg form), and \( b_1 = \cdots = b_{n-1} = 0 \), \( b_n = 1 \). The pair \( (A_0, b_0) \) is obviously controllable. We can easily restate this fact.

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in terms of completions: Let \( A = [a_{ij}]_{i,j=1}^{n} \) be a partial matrix with \( a_{i,i+1} = 1 \) (\( i = 1, \ldots, n - 1 \)), \( a_{ij} = 0 \) if \( j - i \geq 2 \), and all other entries being variables. Then for every completion \( A_c \) of \( A \), the pair \((A_c, b_0)\) is controllable. As an easy corollary we obtain Mirsky's theorem [M]: Given two sets of numbers \( \alpha_1, \ldots, \alpha_n \) and \( \beta_1, \ldots, \beta_n \), there exists a matrix \( A \) with main diagonal \( \alpha_1, \ldots, \alpha_n \) and eigenvalues \( \beta_1, \ldots, \beta_n \) if and only if

\[
\alpha_1 + \cdots + \alpha_n = \beta_1 + \cdots + \beta_n
\]

(indeed, if this equality is satisfied, we take \( A_c \) to be a matrix in the Hessenberg form, and use the Pole-Assignment Theorem to find \( F \) such that \( A_c + b_0 F \) has prescribed eigenvalues; see Lemma 2.3 in Section 2).

This simple but important example serves as a motivation for the present work. We consider partial upper triangular matrices \( A_0 = [a_{ij}]_{i,j=1}^{n}, \) i.e., such that \( a_{ij} \) are specified for \( i \leq j \), and \( a_{ij} \) are variables for \( i > j \) (note that the diagonal elements \( a_{11}, \ldots, a_{nn} \) are given). Such a matrix \( A \) is called irreducible if all "northeastern" rectangular submatrices of sizes \( k \times (n - k) \) (\( k = 1, \ldots, n - 1 \)) are nonzero; in other words,

\[
[a_{ij}]_{1 \leq i \leq k \atop k+1 \leq j \leq n} \neq 0 \quad (k = 1, \ldots, n - 1).
\]

This concept plays a crucial role in the eigenvalue assignment problems of partial upper triangular matrices (see [BGRS] and [RS]).

A direct graph can be associated with a partial upper triangular matrix \( A_0 \) in a natural way. The vertices of the graph are \( \{1, \ldots, n\} \), and there is an edge \( j \to i \) if and only if either \( i > j \) or \( i \leq j \) and \( a_{ij} \neq 0 \).

We now state one of the main results of this paper.

**Theorem 1.1.** The following statements are equivalent for a given \( n \times n \) partial upper triangular matrix \( A \):

(i) \( A \) is irreducible.

(ii) The associated graph of \( A \) is connected, i.e., for any two vertices \( i, j \) there is a directed path from \( i \) to \( j \).

(iii) There exists a completion \( A_c \) of \( A \) such that the pair \((A_c, e_n)\), where

\[
e_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}
\]

is controllable.

(iv) for every \( b \in \mathbb{F}^n, b \neq 0 \), there exists a completion \( A_c \) of \( A \) such that the pair \((A_c, b)\) is controllable.

The proof of this result is given in the next section.

The connectivity of the associated graph of \( A \) can be interpreted informally as structural, or potential, controllability of a pair \((A, e_n)\). Thus, the equivalence of (ii) and (iii) in Theorem 1.1 means that structural controllability is equivalent to controllability.