ON SMALL SOLUTIONS OF DELAY EQUATIONS IN INFINITE DIMENSIONS

S.Z. HUANG$^{1,2}$ and J.M.A.M. van NEERVEN$^{1,3}$

Let $X$ be a Banach space and $1 \leq p < \infty$. Let $L$ be a bounded linear operator from $L^p([-1,0], X)$ into $X$. Consider the delay differential equation

$$\begin{cases}
\dot{u}(t) = Lu_t, & t \geq 0, \\
u(0) = x, & u_0 = f
\end{cases}$$

on the state space $L^p([-1,0], X)$. We prove that a mild solution $u(t) = u(t; x, f)$ is a small solution if and only if the Laplace transform of $u(t; x, f)$ extends to an entire function. The same result holds for the state space $C([-1,0], X)$.

Let $X$ be a Banach space and $1 \leq p < \infty$. For a bounded linear operator $L$ from $L^p([-1,0], X)$ into $X$, on the state space $L^p([-1,0], X)$ we consider the delay differential equation

$$(DDE) \quad \begin{cases}
\dot{u}(t) = Lu_t, & t \geq 0, \\
u(0) = x, & u_0 = f
\end{cases}$$

Here, for a function $u \in L^p_{loc}([-1, \infty), X)$, the functions $u_t \in L^p([-1,0], X)$ are defined by $u_t(s) := u(t+s), t \geq 0, -1 \leq s \leq 0$, and $f \in L^p([-1,0], X)$ is a given 'history' function. A mild solution is a function $u \in L^p([-1, \infty), X)$ such that $u(s) = f(s)$ for $-1 \leq s < 0$ and

$$u(t) = x + \int_0^t Lu_s ds, \quad t \geq 0.$$ 

A mild solution $u(t)$ of (DDE) is called a small solution if $\|u(t)\|$ decays to zero faster than any exponential.

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For the theory of delay functional differential equations in finite-dimensional spaces $X = \mathbb{R}^n$ we refer to the books [3] and [5]. Delay equations in infinite-dimensional spaces have been considered, e.g., in [1], [2], [4], [8], [10], [12], [13].

The purpose of this paper is to prove that a mild solution $u(t)$ of (DDE) is a small solution if and only if its Laplace transform extends to an entire function. In Hilbert space this is an easy result, but the general case where $X$ is allowed to be an arbitrary Banach space depends on the following individual stability theorem for $C_0$-semigroups [11].

**Proposition 1.** Let $T$ be a $C_0$-semigroup on a Banach space $X$, with generator $A$. Let $x_0 \in X$ be such that the local resolvent $\lambda \mapsto R(\lambda, A)x_0$ admits a bounded holomorphic extension to the open right half-plane $\{\text{Re} \lambda > 0\}$. Then for every $\lambda_0 \in \rho(A)$ there exists a constant $M > 0$ such that

$$\|T(t)R(\lambda_0, A)x_0\| \leq M(1 + t) \quad \text{for all} \quad t \geq 0.$$ 

Here, as usual, $\rho(A)$ denotes the set of all $\lambda \in \mathbb{C}$ such that the resolvent $R(\lambda, A) := (\lambda - A)^{-1}$ exists as a bounded linear operator on $X$. An improvement of this result for $B$-convex spaces is given in [7].

Let $1 \leq p < \infty$ and let $L$ be a bounded linear operator from $L^p([-1, 0], X)$ into $X$. In order to treat the problem (DDE) by semigroup methods, we consider the following first order Cauchy problem on the product space $\mathcal{X} := X \times L^p([-1, 0], X)$ (cf. [2], [4] and [8]):

\[
\begin{align*}
\frac{d}{dt} \begin{pmatrix} v(t) \\ w(t) \end{pmatrix} &= A \begin{pmatrix} v(t) \\ w(t) \end{pmatrix} \\
\begin{pmatrix} v(0) \\ w(0) \end{pmatrix} &= \begin{pmatrix} x \\ f \end{pmatrix},
\end{align*}
\]

where the operator matrix $A$ with domain

$$D(A) := \left\{ \begin{pmatrix} x \\ f \end{pmatrix} \in X \times W^{1,p}([-1, 0], X) : x = f(0) \right\}$$

is defined by

$$A \begin{pmatrix} f(0) \\ f' \end{pmatrix} := \begin{pmatrix} Lf \\ f' \end{pmatrix}, \quad f \in W^{1,p}([-1, 0], X).$$

As shown in [2] (see also [4] and [8]), $A$ generates a $C_0$-semigroup $T$ on $\mathcal{X}$. It is easy to see that if $u(t) = u(t; x, f)$ is a mild solution of (DDE) then $\begin{pmatrix} u(t) \\ u_t \end{pmatrix}$ is a mild solution of (ACP) with initial value $\begin{pmatrix} x \\ f \end{pmatrix}$, i.e. we have

$$\begin{pmatrix} u(t) \\ u_t \end{pmatrix} = T(t) \begin{pmatrix} x \\ f \end{pmatrix}. \quad (1)$$