Convergence of Gradient Curves on Hilbert Manifolds*

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Introduction.

Let $M$ be a complete Riemannian Hilbert manifold and $f$ a real valued function on $M$, bounded below. We shall be satisfied throughout with differentiability class $C^3$ for $M$ and $f$. Denote by $G$ the gradient field of $f$. The gradient curve of $f$ starting at some point $x$ in $M$ is the solution of

$$
\frac{d}{dt} \phi(t) = -G(\phi(t)), \quad \phi(0) = x.
$$

The basic and immediate information on the gradient curve is incorporated in:

1. \[ f(\phi(t)) + \int_0^t \|G(\phi(\tau))\|^2 d\tau = f(\phi(0)), \]
2. \[ d(\phi(b), \phi(a)) \leq \int_a^b \|G(\phi(t))\| dt, \]

with $d$ the distance function on $M$. This implies, together with the completeness of $M$, that $\phi(t)$ is defined in $0 \leq t < \infty$ and moreover, we have a sequence $t_n \to \infty$, such that $\|G(\phi(t_n))\|$ converges to zero. The condition (C) of Palais and Smale [5] requires such a sequence $\phi(t_n)$ to have a limit point and the extension of Morse theory to Hilbert manifolds becomes possible. Thus it has not been necessary for Morse theory to settle the question, whether $\phi(t)$ actually converges for $t \to \infty$, which however, may become important in some concrete cases. In this paper we shall give a positive answer to this question under rather mild assumptions on the function $f$. We shall first treat the general case and then come to the specific case of energy of curves, which aroused our interest in the subject.

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Gradient Estimates

We consider the derivative $df$ of $f$ as a continuous linear form on the tangent space $T_x M$ at any point $x$ in $M$. Then by definition of $G$:

$$df(v) = \langle G, v \rangle,$$

for all tangent vectors $v$ in $T_x M$. Making use of the Levi-Civita connection on $M$, determined by the Riemannian metric $\langle \cdot, \cdot \rangle$, we can differentiate $df$ and $G$ covariantly [2] and obtain the relation:

$$Vdf(u, v) = \langle V_u G, v \rangle = \langle Hu, v \rangle,$$

at any point in $M$, where $H$ is a bounded linear operator on $T_x M$ defined by $Hu = V_u G$ and called the Hessian of $f$. Usually $H$ has only been defined at critical points, where it is independent of the connection. We shall use $H$ as a $C^1$ section in $L(TM, TM)$.

$H$ is a self-adjoint operator as $Vdf$ is a symmetric form and we shall assume, that $H$ is a Legendre operator or $\langle Hu, u \rangle$ a Legendre form [4]. Then the spectrum of $H$ below some positive number, consists of finitely many eigenvalues with finite multiplicities. In particular $H$ has finite index and finite nullity at each point of $M$. For some $\kappa \geq 0$ below the positive spectrum of $H$ at some point $x$ in $M$, we shall define $P^0 = P^0(x)$ to be the orthogonal projection of $T_x M$ onto the linear span of all eigenvectors of $H$, with eigenvalues in the interval $[-\kappa, \kappa]$. We shall denote by $P^- = P^-(x)$ the orthogonal projection of $T_x M$ onto the linear span of all eigenvectors of $H$ with eigenvalues $< -\kappa$ and by $P^+$ the orthogonal projection of $T_x M$ onto the orthogonal complement of the sum of the images of $P^0$ and $P^-$. Choosing $\kappa$ smaller than the smallest nonzero eigenvalue at a point $x_0$, we can, since $H$ is continuous, find a neighbourhood $U$ of $x_0$ in $M$, such that

$$\langle H P^+ v, P^+ v \rangle > \mu \|P^+ v\|^2,$$

$$-\kappa \|P^0 v\|^2 \leq \langle H P^0 v, P^0 v \rangle < \kappa \|P^0 v\|^2,$$

$$\langle H P^- v, P^- v \rangle \leq -\mu \|P^- v\|^2$$

holds, with $0 \leq \kappa < \mu$, for all $v \in T_x M$ and $x$ in $U$. If the nullity of $H$ at $x_0$ is null, then we may choose $\kappa = 0$; otherwise we must take $\kappa > 0$. It follows from the spectral theorem of self-adjoint operators, that the orthogonal projections are $C^1$ sections in the bundle of linear operators on tangent spaces of $M$ over the neighbourhood $U$, since $H$ is of class $C^1$ with its spectrum in the three disjoint intervals mentioned above. We shall call a neighbourhood $U$ of some point $x_0$ in $M$, having the above properties, a regular $\kappa, \mu$ neighbourhood.