This paper investigates the minimal degree of polynomials $f \in \mathbb{R}[x]$ that take exactly two values on a given range of integers $\{0, \ldots, n\}$. We show that the gap, defined as $n - \deg(f)$, is $O(n^{\frac{1}{5}48})$. The maximal gap for $n \leq 128$ is 3. As an application, we obtain a bound on the Fourier degree of symmetric Boolean functions.

1. Introduction

We consider polynomials $f \in \mathbb{R}[x]$ that take only two values on the domain $\{0, \ldots, n\}$. For each $n$, we ask how small the degree of such a (nonconstant) polynomial can be. We may assume without loss of generality that the range of $f$ is $\{0,1\}$. Since arbitrary prescribed values can be interpolated by a polynomial of degree at most $n$, we may assume that $\deg(f) \leq n$. Then $f$ is uniquely determined, and $f \in \mathbb{Q}[x]$. We seek bounds on the gap between $\deg(f)$ and $n$.

More precisely, for any $n \geq 1$ and $a = (a_0, \ldots, a_n) = (f(0), \ldots, f(n)) \in \mathbb{Q}^{n+1}$, with $f \in \mathbb{Q}[x]$ of degree at most $n$, we define the gap of $a$ as

$$\gamma(a) = n - \deg(f),$$

where the degree of any constant polynomial is taken to be zero. For $n \geq 1$, the maximal gap is

$$\Gamma(n) = \max_{a \in A_n} \gamma(a),$$

where

$$A_n = \{0,1\}^{n+1} \setminus \{(0, \ldots, 0), (1, \ldots, 1)\}.$$  

As an example, if $a = (1,0, \ldots, 0)$, then $f$ has at least $n$ zeros (at 1,2, \ldots, $n$) and thus has degree at least $n$. But $\deg(f) \leq n$, so that $\deg(f) = n$ and $\gamma(a) = 0$. 

Mathematics Subject Classification (1991): 68R05; 11B83, 11Y50, 11B39
In Section 2 we characterize the property \( \gamma(a) \geq r \) by \( r \) linear equations in \( a_0, \ldots, a_n \). Then we exhibit a family of vectors with gap one and note that only an exponentially small fraction of vectors has positive gap. The upper bound \( \Gamma(n) < n/2 \) is trivial. We show that \( \Gamma(n) = 0 \) if \( n+1 \) is prime; using a bound on the gap between consecutive primes, we then obtain \( \Gamma(n) = O(n^{0.548}) \) in general. We conjecture that, in fact, \( \Gamma(n) = O(1) \).

In Section 3 we introduce the notion of a folded vector. The characterization of the gap then leads to several infinite families of vectors, again with gap one, via the solution of certain Diophantine equations. In Section 4 we extend and combine these examples to obtain families with gap two or three, and then give further examples with gap one; the latter do not, however, give any new information about \( \Gamma \). In Section 5 we report on a computer search which determined all vectors with positive gap for \( n \leq 128 \); the largest gap is 3.

This research was motivated by work of Nisan and Szegedy [3]. They investigate the degree of polynomials in \( \mathbb{R}[x_1, \ldots, x_n] \) that interpolate (or approximate) a given Boolean function \( g : \{0,1\}^n \to \{0,1\} \). The smallest degree of such interpolating polynomials is the Fourier degree of \( g \). If \( g \) is symmetric, there is an associated function \( f : \{0, \ldots, n\} \to \{0,1\} \) whose interpolation problem is equivalent to the original one. Bounds on \( \Gamma(n) \) are thus equivalent to bounds on the Fourier degree of symmetric Boolean functions.

## 2. Bounds on the Gap

We begin by recalling a few basic facts from the theory of difference equations; see Graham, Knuth, and Patashnik [1] for a more complete discussion. Given \( g \in \mathbb{R}[x] \), we define its discrete derivative \( Dg \) by

\[
(Dg)(x) = (D^1g)(x) = g(x) - g(x - 1).
\]

For \( i \geq 2 \), we define the discrete derivative \( D^i g \) of order \( i \) inductively by

\[
D^i g = D(D^{i-1}g).
\]

**Proposition 2.1.** For \( g \in \mathbb{R}[x] \) and \( m \geq 1 \), the following hold:

(i) \( (D^m g)(x) = \sum_{0 \leq j \leq m} (-1)^j \binom{m}{j} g(x - j) \).

(ii) If \( \deg(g) = m \), then \( \deg(Dg) = m - 1 \).

(iii) If \( g \) is constant, then \( Dg = 0 \).

(iv) If \( \deg(g) = m \), then \( D^m g \) is a nonzero constant.

(v) \( D^m g = 0 \iff \deg(g) < m \).

**Proof.** (i) follows by induction on \( m \); (ii) and (iii) follow immediately when \( g \) is written out as a sum of monomials; and (iv) and (v) follow from (ii) and (iii).

Now we characterize the gap \( \gamma(a) \) in terms of binomial sums.