We show that if a graph of $v$ vertices can be drawn in the plane so that every edge crosses at most $k > 0$ others, then its number of edges cannot exceed $4.108 \sqrt{kv}$. For $k \leq 4$, we establish a better bound, $(k+3)(v-2)$, which is tight for $k = 1$ and $2$. We apply these estimates to improve a result of Ajtai et al. and Leighton, providing a general lower bound for the crossing number of a graph in terms of its number of vertices and edges.

1. Introduction

Given a simple graph $G$, let $v(G)$ and $e(G)$ denote its number of vertices and edges, respectively. We say that $G$ is drawn in the plane if its vertices are represented by distinct points of the plane and its edges are represented by Jordan arcs connecting the corresponding point pairs but not passing through any other vertex. Throughout this paper, we only consider drawings with the property that any two arcs have at most one point in common. This is either a common endpoint or a common interior point where the two arcs properly cross each other. We will not make any notational distinction between vertices of $G$ and the corresponding points in the plane, or between edges of $G$ and the corresponding Jordan arcs.

We address the following question. What is the maximum number of edges that a simple graph of $v$ vertices can have if it can be drawn in the plane so that every edge crosses at most $k$ others? For $k = 0$, i.e. for planar graphs, the answer is $3v - 6$. Our first theorem generalizes this result to $k \leq 4$. The case $k = 1$ has been discovered independently by Bernd Gärtner, Torsten Thiele, and Günter Ziegler (personal communication).
Theorem 1. Let $G$ be a simple graph drawn in the plane so that every edge is crossed by at most $k$ others. If $0 \leq k \leq 4$, then we have

$$e(G) \leq (k + 3)(v(G) - 2).$$

For $k = 0, 1, 2$, the above bound cannot be improved (see Remark 2.3 at the end of the next section.)

The crossing number $cr(G)$ of a graph $G$ is the minimum number of crossing pairs of edges, over all drawings of $G$ in the plane.

Ajtai et al. [1] and, independently, Leighton [4] obtained a general lower bound for the crossing number of a graph, which found many applications in combinatorial geometry and in VLSI design (see [5], [6], [8]). Our next result, whose proof is based on Theorem 1, improves the bound of Ajtai et al. by roughly a factor of 2.

Theorem 2. The crossing number of any simple graph $G$ satisfies

$$cr(G) \geq \frac{1}{33.75} e^3(G) - 0.9v(G) > 0.029 \frac{e^3(G)}{v^2(G)} - 0.9v(G).$$

Theorem 3. Let $G$ be a simple graph drawn in the plane so that every edge is crossed by at most $k$ others, for some $k \geq 1$. Then we have

$$e(G) \leq \sqrt{16.875kv(G)} \approx 4.108 \sqrt{kv(G)}.$$

Theorems 2 and 3 do not remain true if we replace the constants 0.029 and 4.108 by 0.06 and 1.92, respectively (see Remarks 3.2 and 3.3).

In the last section, we use the ideas of Székely [8] to deduce some consequences of Theorem 2.

2. Proof of Theorem 1

First we need a lemma for multigraphs, i.e., for graphs that may have multiple edges. In a drawing of a multigraph, any two non-disjoint edges either share only endpoints or have precisely one point in common, at which they properly cross.

Let $M$ be a multigraph drawn in the plane so that every edge crosses at most $k$ other edges. Let $M'$ be a sub-multigraph of $M$ with the largest number of edges such that in the drawing of $M'$ (inherited from the drawing of $M$), no two edges cross each other. We say that $M'$ is a maximum plane sub-multigraph of $M$, and its faces will be denoted by $\Phi_1, \Phi_2, \ldots, \Phi_m$. Let $|\Phi_i|$ denote the number of edges of $M'$ along the boundary of $\Phi_i$, where every edge whose both sides belong to the interior of $\Phi_i$ is counted twice. It follows from the maximality of $M'$ that every edge $e$ of $M$ which does not belong to $M'$ (in short $e \in M - M'$) crosses at least one edge of $M'$. The closed portion between an endpoint of $e$ and the nearest crossing