Translation Group and Spectrum Condition

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Abstract. Let \( \{A, \mathbb{R}^d, \alpha\} \) be a C*-dynamical system, where \( \mathbb{R}^d \) is the d-dimensional vector group. Let \( V \) be a convex cone in \( \mathbb{R}^d \) and \( \hat{V} \) its dual cone. We will characterize those representations of \( A \) with the properties (i) \( \alpha_a, a \in \mathbb{R}^d \) is weakly inner, (ii) the corresponding unitary representation \( U(a) \) is continuous, and (iii) the spectrum of \( U(a) \) is contained in \( \hat{V} \).

I. Introduction

The spectrum condition is one of the essential ingredients of quantum field theory. Especially the discovery of the fact that the translations are weakly inner automorphisms for finite particle representations [4] has made the spectrum condition an interesting subject. Many problems in connection with this have been studied and answered [4–7]. In the previous investigations, which are based on the “covariance-algebra” introduced by Doplicher, Kastler, and Robinson [9], it has been assumed that the translation group is acting strongly continuous on the C*-algebra in question. On the other hand, in the theory of local observables, one usually is only interested in representations which are locally normal with respect to the vacuum representation. But this means that the algebra associated to a bounded region should be a von Neumann algebra. Such an assumption, however, contradicts the assumption of strong continuity of the translations. Since in a recent paper [7] it has been shown that one can handle the problem of covariant representation without using the continuity of the group action on the algebra, we will treat the problem of the spectrum condition again.

Furthermore in the existing literature only the one dimensional case and its iterations have been treated with full mathematical rigour. But the case where the cone in question is an arbitrary convex cone with interior points is still missing. We also want to fill this gap.

In the next section we handle the one dimensional case again. We show that by introducing the reasonable concepts one can reduce this problem to results existing in the literature. The results obtained here are generalized in Sect. III to the n-dimensional case where the spectrum is restricted to a half space. The n-dimensional case where the spectrum is in a cone is treated in Sect. IV and V.
II. The One Dimensional Case

Let $A$ be a $C^*$-algebra and $G$ be a topological group, acting as a group of automorphisms on $A$, i.e. $\alpha: G \to \text{Aut}(A)$. Following [7] we will denote:

$$A^*_c = \{ \phi \in A^*; \ g \to \phi \circ \alpha_g \text{ is a continuous function on } G \text{ with values in the Banach space } A^* \}.$$  

This is a norm closed linear sub-space of $A^*$, invariant under the transposed action $\alpha^*_g$ and generated by its positive elements.

If $G$ is a locally compact group with left invariant Haar measure $dg$, then for $\phi \in A^*_c, f \in L^1(G), \ x \in A^{**}$, the expression $\int \phi(\alpha_g x) f(g) \, dg$ is well defined and defines a continuous linear functional on $A^*_c$. The set of all continuous linear extensions to all of $A^*$ will be denoted by $[x(f)]$. If $y \in [x(f)]$, then we have $[x(f)] = y + N_c$, where $N_c$ is the annihilator of $A^*_c$ in $A^{**}$.

Having these notations at hand, we can use, in the case where $G$ is also abelian, the spectral theory of Arveson [1] ($\alpha^*_g$ acts strongly continuous on $A^*_c$, and $A^{**}/N_c$ is the dual space of $A^*_c$), and the results obtained from it by linear methods. Using the notations of G. K. Pedersen [11, Chap. 8] we define for $G = \mathbb{R}$ the space $R(-\infty, \mu) = A^{**}$ to be the $\sigma(A^{**}, A^*)$ closed linear sub-space generated by all $[x(f)]$ with $x \in A^{**}, f \in L^1(\mathbb{R})$ with $\hat{f}$ having compact support and $\text{supp} \hat{f} \subset (-\infty, \mu)$. ($\hat{f}$ denotes the Fourier-transform of $f$.)

In the same manner as in the case where $\alpha^*_g$ acts strongly continuous on $A$, we define $E(\lambda) = \text{projection onto the common null space of all } y \in R(-\infty, -\lambda)$, i.e. $E(\lambda) = \text{maximal projection } E \text{ in } A^{**}$ such that $R(-\infty, -\lambda) E = 0$ and $E(\infty) = \text{s-lim } E(\lambda)$. Our aim is to show that these projections have the same properties as the corresponding projections one obtains when $\alpha_g$ acts strongly continuous on $A$. In order to show this we define:

II.1. Definition. Let $\{A, \mathbb{R}, \alpha\}$ be a $C^*$-dynamical system, and let $E(\lambda)$ be the projections defined above. Then we denote:

(a) $A^*(\mathbb{R}^+) = \{ \phi \in A^*; E(\infty) \phi = \phi E(\infty) = \phi \}$,
(b) $A^*_0(\mathbb{R}^+) = \{ \phi \in A^*; \text{ such that there exist } \lambda, \mu < \infty \text{ with } E(\mu) \phi = \phi E(\lambda) = \phi \}.$

II.2. Proposition. With the above notation we obtain:

(i) $A^*_0(\mathbb{R}^+) \text{ is norm-dense in } A^*(\mathbb{R}^+)$.

(ii) $\phi \in A^*(\mathbb{R}^+)$ and $x, y \in A^{**}$ implies $x \phi y \in A^*(\mathbb{R}^+)$ or equivalently $E(\infty) \text{ belongs to the center of } A^{**}.$

(iii) An element $\phi \in A^*$ belongs to $A^*_0(\mathbb{R}^+)$ if and only if the following conditions are fulfilled:

(a) $a \to \phi(x \alpha_a(y))$ is continuous and it is the boundary value of an analytic function, $W_1(z)$ holomorphic in upper half-plane satisfying the estimate $|W_1(z)| \leq \|x\| \cdot \|y\| \cdot \|\phi\| \cdot \text{exp} \{m|\text{Im } z|\}$ for a suitable constant $m$.

(b) $a \to \phi(\alpha_a(x)y)$ is continuous and it is the boundary value of an analytic function $W_2(z)$ holomorphic in the lower half-plane fulfilling the estimate $|W_2(z)| \leq \|x\| \cdot \|y\| \cdot \|\phi\| \cdot \text{exp} \{m'|\text{Im } z|\}$ with a suitable constant $m'$. 