SMOOTHNESS OF SOLUTIONS OF A NONLINEAR ODE

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Smoothness of a $C^\infty$-function $f$ is measured by (Carleman) sequence $\{M_k\}_0^\infty$, we say $f \in C^\infty_M[0,1]$ if $|f^{(k)}(t)| \leq CR^k M_k$, $k = 0, 1, \ldots$ with $C,R > 0$. A typical statement proven in this paper is: THEOREM: Let $u, b$ be two $C^\infty$-functions on $[0,1]$ such that (a) $u' = u^2 + b$, (b) $|b^{(k)}(t)| \leq CR^k (k!)^\gamma$, $\gamma > 1$, $k \in \mathbb{Z}_+$. Then $|u^{(k)}(t)| \leq C_1 R^k ((k - 1)!)^\gamma$, $k \in \mathbb{Z}_+$.

1 Introduction and setting the problem

1. A general problem in the theory of linear PDE is to determine or describe the smoothness of a solution of

$$Lu = b \quad \text{in } \Omega \subset \mathbb{R}^n,$$

in terms of the smoothness of known right-hand side $b$ and/or boundary data $u|_{\partial \Omega}$. Smoothness of $u$ is measured in terms of special parametrized families of functional spaces (Sobolev, Besov, Carleman, Gevrey classes, defined by using weights for $f$ or its Fourier transform). These so called a priori estimates guarantee nice differential properties of a solution if we know that it does exist. But moreover, these a priori estimates (for $L$ or its dual $L'$) often guarantee [by duality principle] the existence of a solution and its uniqueness as well. In the case of non-linear equations, even in dimension 1, that is ODE, blowups happen within a finite time period, and smoothness by itself is of no help.

2. A simple example shows our problems. Let us consider an equation

$$u' = u^2 \quad \text{on } [0,1], \quad u(0) = a \neq 0. \quad (1.1)$$

The solution exists in some neighborhood of 0,

$$u(t) = \frac{1}{A - t}, \quad A = 1/a,$$

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but if $a > 1$ it blows up at $A$, $0 < A < 1$, and there is no solution for (1.1). But

$$b = u' - u^2$$

is identically zero. What could be smoother?

We do not try to find conditions on a function $b$ and a constant $a$ which would guarantee the existence of a solution of the equation

$$u' = u^2 + b, \quad 0 \leq t \leq 1, \quad u(0) = a,$$

on $I = [0, 1]$. We accept (1.2) as a given relationship between two ($C^\infty$-) functions, and under the assumption

$$|b^{(k)}(t)| \leq CR^k M_k, \quad k = 0, 1, \ldots ; \quad t \in I$$

(1.3)

we want to know how smooth $u$ could (should) be. In particular, do (1.3) and (1.2) imply that

$$|u^{(k)}(t)| \leq C_1 R_1^k M_k, \quad k = 0, 1, \ldots ; \quad t \in I?$$

(1.4)

Could we choose $R_1$ depending on $R$ but not on $C$? Could we choose

$$R_1 = R?$$

We give positive answers to all these questions in the case of Gevrey-Carleman classes with

$$M_k = (k!)^\gamma, \quad \gamma > 1,$$

(1.5)

see Theorem 1 in Section 2; it is a central result of this paper.

3. If

$$M_k = (k!)^\alpha, \quad \alpha < 1,$$

(1.6)

the answers are negative as we can easily see in the case (1.1), i.e. $b = 0$. With necessity

$$u(t) = (A - t)^{-1} \quad \text{for some} \quad A \notin I,$$

but this function does not satisfy the system of inequalities

$$|u^{(k)}(t)| \leq C_1 R_1^k (k!)^\alpha, \quad \alpha < 1,$$

whatever our choice of $R_1$ would be. Indeed,

$$u^{(k)}(t) = k!(A - t)^{-(k+1)}$$

and with

$$d = \min_{0 \leq t \leq 1} |A - t| > 0, \quad D = 1/d,$$