A New Proof of Localization in the Anderson Tight Binding Model

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Abstract. We give a new proof of exponential localization in the Anderson tight binding model which uses many ideas of the Frohlich, Martinelli, Scoppola and Spencer proof, but is technically simpler—particularly the probabilistic estimates.

1. Introduction

The Anderson tight binding model is given by the random Hamiltonian $H = -\Delta + V$ on $l^2(\mathbb{Z}^d)$, where $\Delta(x,y) = 1$ if $|x-y| = 1$ and zero otherwise, and $V(x), x \in \mathbb{Z}^d$, are independent identically distributed random variables with common probability distribution $\mu$. This model was introduced by Anderson [1] to describe the motion of a quantum-mechanical electron in a crystal with impurities.

It is well known that the spectrum of the Hamiltonian $H$ is given by

$$\sigma(H) = \sigma(-\Delta) + \sigma(V) = [-2d, 2d] + \text{supp } \mu$$

with probability one [2, 3]. The spectrum of $H$ can be decomposed into pure point spectrum, $\sigma_{\text{pp}}(H)$, absolutely continuous spectrum, $\sigma_{\text{ac}}(H)$, and singular continuous spectrum, $\sigma_{\text{sc}}(H)$. There exist sets $\Sigma_{\text{pp}}, \Sigma_{\text{ac}}, \Sigma_{\text{sc}} \subset \mathbb{R}$ such that $\sigma_{\text{pp}}(H) = \Sigma_{\text{pp}}, \sigma_{\text{ac}}(H) = \Sigma_{\text{ac}}$ and $\sigma_{\text{sc}}(H) = \Sigma_{\text{sc}}$ with probability one [3].

In this article we are concerned with localization. We say that the random operator $H$ exhibits localization in an energy interval $I$ if $H$ has pure point spectrum in $I$ with probability one, i.e., if $\Sigma_{\text{ac}} \cap I = \Sigma_{\text{sc}} \cap I = \emptyset$. We have exponential localization in $I$ if we have localization and all the eigenfunctions corresponding to eigenvalues in $I$ have exponential decay.

Exponential localization for the Anderson tight binding Hamiltonian is well understood in one dimension [3–6], where it was first established in the continuum by Gol’dsheid, Molchanov and Pastur [20]. In higher dimensions, the first results toward localization, for either high disorder or low energy, were due to Fröhlich...
and Spencer [7], who proved exponential decay for the Green's functions. These were followed by a proof of localization for a hierarchical version of $H$ by Jona-Lasinio, Martinelli and Scoppola [8], and by a proof of the absence of absolutely continuous spectrum at higher disorder or low energy by Martinelli and Scoppola [9]. Subsequently, proofs of exponential localization, at high disorder or low energy, were given by Fröhlich, Martinelli, Scoppola and Spencer [4], Delyon, Levy and Souillard [10], and Simon and Wolff [11]. All of these higher dimensional results relied on methods or results of [7].

Recently, von Dreifus and Spencer [12, 13] introduced a new proof of the original Fröhlich and Spencer results in [7], which uses the same basic ideas, but is technically much simpler—particularly the probabilistic estimates. The key new idea is a scaling argument previously used in the study of bond percolation [14].

In this article we show how the methods of von Dreifus and Spencer can be used to give a direct proof of exponential localization. This proof uses the basic ideas behind the Fröhlich, Martinelli, Scoppola and Spencer proof [4, 5], but has much simpler probabilistic estimates. As in [6], we can allow singular distributions for the potential not permitted in [10, 11].

This article is organized as follows: We state our results in Sect. 2. Theorem 2.1 is our result on localization, it follows from Theorems 2.2 and 2.3. Theorem 2.2 is our basic technical result. Theorems 2.3 and 2.2 are proved in Sects. 3 and 4, respectively. The Appendix contains a discussion of when the hypotheses of Theorem 2.1 can be proven so we can conclude localization.

2. Statement of Results

We start with some notations and definitions.

If $A \subset \mathbb{Z}^d$, we denote by $H_A$ the operator $H$ restricted to $L^2(A)$ with zero boundary conditions outside $A$. The corresponding Green's function is $G_A(z) = (H_A - z)^{-1}$, defined for $z \notin \sigma(H_A)$. We will write $G_A(z; x, y) = (H_A - z)^{-1}(x, y)$ for $x, y \in A$.

If $A = \mathbb{Z}^d$ we simply write $G(z; x, y)$. Notice that we omit the dependence of $H_A$ and $G_A$ on the potential $V$.

If $x \in \mathbb{Z}^d$, $x = (x_1, \ldots, x_d)$, let $\|x\| = \max \{|x_1|, \ldots, |x_d|\}$. It will be convenient to use this norm in $\mathbb{Z}^d$. The distances in $\mathbb{Z}^d$ will always be taken with respect to this norm.

If $L > 0$, $x \in \mathbb{Z}^d$, we will denote by $A_L(x)$ the cube centered at $x$ with sides of length $L$, i.e.,

$$A_L(x) = \left\{ y \in \mathbb{Z}^d; \|y - x\| \leq \frac{L}{2} \right\}.$$ 

By $\partial A_L(x)$ we will denote its boundary, i.e.,

$$\partial A_L(x) = \left\{ \langle y, y' \rangle; y \in A_L(x), y' \notin A_L(x), \sum_{i=1}^d |y_i - y'_i| = 1 \right\}.$$ 

We will abuse the notation and write $y \in \partial A_L(x)$ to mean $\langle y, y' \rangle \in \partial A_L(x)$ for some