ON BLOCKING SETS OF QUADRICS

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We determine the three smallest blocking sets with respect to lines of the quadric $Q(2n, q)$ with $n \geq 3$ and the two smallest blocking sets with respect to lines of the quadric $Q^+(2n+1, q)$ with $n \geq 2$. These results will be used in a forthcoming paper for determining the smallest blocking sets with respect to higher dimensional subspaces in the quadrics $Q(2n, q)$ and $Q^+(2n+1, q)$.

1 INTRODUCTION

One possibility to obtain insight in the structure of a geometry is to study substructures with certain properties. One of these substructures are blocking sets, which have been studied quite intensively in projective spaces [1]. Usually one is particularly interested in the smallest blocking sets.

In this paper, we study blocking sets of quadrics of hyperbolic type $Q^+(2n + 1, q)$ and parabolic type $Q(2n, q)$. For a definition and information on quadrics, see [6]. Here we recall only that a quadric consists of a set of points of a projective space and of all subspaces of a projective space contained in this set. The point set is defined by a non-degenerate quadratic form of the underlying vector space.

By a blocking set of a quadric with respect to subspaces of dimension $s$ we mean a set $B$ of points that has the property that every subspace of dimension $s$ meets $B$ in at least one point. If $s = 1$, we simply call $B$ a blocking set.

It depends on the value of $s$ whether it is difficult or not to determine the smallest blocking set with respect to subspaces of dimension $s$ in the quadric $Q^+(2n + 1, q)$. In fact, for $s = 1$ and $s = 2$, the problem is easy and has been solved in [8]. If $s = n$, so that the subspaces of dimension $s$ are the maximal subspaces of $Q^+(2n + 1, q)$, then the problem seems to be very difficult. In fact, it is even not known in general, if $Q^+(2n + 1, q)$ admits an ovoid, that is a set of points that meets every maximal subspace of $Q^+(2n + 1, q)$ in exactly one
For $2 < s < n$, it seems likely that the answer can be found by induction on $s$. For such a proof, it would be extremely helpful not to know only the smallest blocking sets with respect to subspaces of a certain dimension but also all blocking sets that are slightly bigger. The reason is the following. In an inductive proof one will consider blocking sets that are induced in subspaces or in quotient geometries. But only the average number of induced points can be controlled and this average number usually will not correspond to the minimum size of points in a blocking set for the induced structure. Therefore we will determine also the second and, for $Q(2n, q)$, the third smallest minimal blocking set of $Q(2n, q)$ and $Q^+(2n + 1, q)$. These will be all minimal blocking sets that are contained in a hyperplane of the corresponding projective space.

In a forthcoming paper, the results obtained will be applied to solve problems of smallest blocking sets with respect to higher dimensional subspaces.

Consider the quadric $Q = Q^+(2n + 1, q)$ defined by a quadratic form in $PG(2n + 1, q)$. Since a hyperplane $H$ of $PG(2n + 1, q)$ meets every line of $PG(2n + 1, q)$, it is clear that $H \cap Q$ meets every line of $Q$. However, there are two different types of hyperplanes, those which are tangent to $Q$ and those which are not. A non-tangent hyperplane $H$ meets $Q$ in $|Q(2n, q)| = (q^{2n} - 1)/(q - 1)$ points and, since every point of $H \cap Q$ lies on lines of $Q$ that are not contained in $Q$, the set $H \cap Q$ is a minimal blocking set. A the tangent hyperplane $H$ meets $Q$ in $1 + q|Q^+(2n - 1, q)| = (q^{2n} - 1)/(q - 1) + q^{n-1} + 1$ points, but the set $H \cap B$ is not a minimal blocking set, the pole of the hyperplane $H$ must be removed to obtain a minimal blocking set. In fact, these two examples provide the two smallest minimal blocking sets as the following first result of this paper shows.

**Theorem 1.1** Let $B$ be a blocking set of $Q^+(2n + 1, q) \subseteq PG(2n + 1, q)$ where $n \geq 3$. If $n = 2$ suppose that $|B| \leq q|Q^+(2n - 1, q)|$. If $n \geq 3$ suppose that $|B| \leq 1 + q|Q^+(2n - 1, q)|$. Then $B$ contains a blocking set that is contained in a hyperplane of $PG(2n + 1, q)$. In particular $|B| \geq |Q(2n, q)|$.

For polar spaces $Q$ of type $Q(2n, q)$, the situation is as follows. The smallest size of a blocking set with respect to hyperplanes is known only in the cases $s = 1$ [8] and $s = n - 2$ [4]. Non-tangent hyperplanes meet $Q$ either in a $Q^-(2n - 1, q)$ or in a $Q^+(2n + 1, q)$, tangent hyperplanes meet $Q$ in a cone over a $Q(2n - 2, q)$. The intersection of $Q$ with a non-tangent hyperplane gives a minimal blocking set, the intersection with a tangent hyperplanes $H$ is again not minimal, the pole of $H$ has to be removed. The three minimal blocking sets obtained in this way have respectively $|Q^-(2n - 1, q)| = u - q^{n-1}$, and $|Q^+(2n - 1, q)| = u + q^{n-1}$, and $|Q(2n - 2, q)| = u - 1$ points, where $u = (q^{2n-1} - 1)/(q - 1)$. The second result of this paper is the following.

**Theorem 1.2** Let $B$ be a blocking set of $Q(2n, q) \subseteq PG(2n, q)$ where $n \geq 3$. If $(n, q) = (3, 2)$ suppose that $|B| \leq |Q^+(2n - 1, q)| - 1$. If $(n, q) \neq (3, 2)$ suppose that $|B| \leq |Q^+(2n - 1, q)|$. Then $B$ contains a blocking set that is contained in a hyperplane of $PG(2n, q)$. In particular $|B| \geq |Q^-(2n - 1, q)|$.

It is unlikely that the bounds of these two theorems are best possible. In $Q(2n, q)$, an example of a small blocking set that does not contain a blocking set in a hyperplane can