A characterization of Wiener's algebra on locally compact groups

By

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It is the purpose of this paper to show that the algebra $W(\mathbb{R})$, which has first been defined by N. Wiener, is the smallest among all Segal algebras on $\mathbb{R}$ which are at the same time $C^0(\mathbb{R})$-modules with respect to pointwise multiplication. By means of a new characterization of $W(\mathbb{R})$ this result can be extended to general locally compact groups.

Our notation follows that of [2] and [3]. $G$ denotes a locally compact group. In order to avoid trivialities we assume that $G$ is nondiscrete. $K(G)$ shall denote the space of all continuous functions $k$ with compact support ($\text{supp } k$). The Banach space $C^0(G)$ consists of all continuous functions on $G$ vanishing at infinity. It is the closure of $K(G)$ with respect to the norm $\|f\|_\infty = \sup_{x \in G} |f(x)|$. The Haar measure of a measurable set $M \subseteq G$ shall be denoted by $|M|$. For $y \in G$ the left and right translation operators are defined by $L_y f(x) = f(y^{-1} x)$ and $R_y f(x) = f(xy^{-1}) \Delta^{-1}(y)$, $\Delta$ being the Haar module on $G$. For facts concerning Segal algebras the reader is referred to [2] and [3]. A Segal algebra $S(G)$ is a Banach module over $C^0(G)$ (with respect to multiplication), if $g$ is in $S(G)$ for every $f \in S(G)$ and $g \in C^0$ and satisfies $\|fg\|_S \leq \|f\|_S \cdot \|g\|_\infty$. Finally let us recall that Wiener’s algebra $W(\mathbb{R})$ consists of all continuous functions $f$ on $\mathbb{R}$ satisfying

$$\|f\|_W = \sum_{n \in \mathbb{Z}} \max_{0 \leq x \leq 1} |f(n + x)| < \infty$$

(cf. [2], Chap. I, § 5, ex. iii). With the norm

$$\|f\|_W = \sup_{y \in \mathbb{R}} \|L_y f\|_W$$

$W(\mathbb{R})$ is a Segal algebra which is of course a $C^0(\mathbb{R})$-module.

**Theorem 1.** Let $S(\mathbb{R})$ be a Segal algebra on $\mathbb{R}$ which is a $C^0(\mathbb{R})$-module. Then $S(\mathbb{R})$ contains Wiener’s algebra $W(\mathbb{R})$ and $\|f\|_S \leq C \|f\|_W$ holds for all $f \in W(\mathbb{R})$. Hence $W(\mathbb{R})$ is the smallest among all Segal algebras which are $C^0(\mathbb{R})$-modules.

**Proof.** Let $f \neq 0, f \in S(\mathbb{R})$ be given. Then there is some $k_1 \in K(\mathbb{R})$ with $f_1 := k_1 \cdot f \neq 0$, in particular $f_1$ is a continuous function in $S(\mathbb{R})$. Let $U$ be an open set such that $|f_1(x)| \neq 0$ for all $x \in U$. Then there is some $h \in K(\mathbb{R})$, $h \neq 0$ with $0 \leq h(x) \leq 1$ and $\text{supp } h \subseteq U$. Consequently $f_2 := |f_1| \cdot h = f_1 \cdot h |f_1|/f_1$ is in $S(\mathbb{R})$, $h |f_1|/f_1$ being in
Moreover \( f_2 \) is positive. Thus we may choose an open set \( V \subseteq U \) with \( f_2(x) \geq \delta > 0 \) for all \( x \in V \). Let now \( k \in K(\mathbb{R}) \) be given. Without loss of generality we may suppose that \( k \) is positive. Then there is a finite sequence \( (y_i)_{i=1}^n \subseteq \mathbb{R} \) with \( \text{supp} \, k = \bigcup_{i=1}^n y_i + V \). Hence
\[ k(x) \leq \|k\|_\infty \delta^{-1} \sum_{i=1}^n L_{y_i} f_2(x) := f_3(x) \quad \text{for all } x \in \mathbb{R}. \]
This implies that \( k/f_3 \) is in \( K(\mathbb{R}) \) and that \( k = f_3 \cdot k/f_3 \) is in \( S(\mathbb{R}) \). In particular \( S(\mathbb{R}) \) contains a function \( g \in K(\mathbb{R}) \) with \( 0 \leq g(x) \leq 1 \) and \( g(x) \equiv 1 \) on \([0, 1]\). Let now \( f \in W(\mathbb{R}) \) be given. Then with \( a_n(f) = \max_{0 \leq x \leq 1} |f(n + x)| \) we have for every \( x \in \mathbb{R} \)
\[ |f(x)| \leq \sum_{n \in \mathbb{Z}} a_n(f) L_n g(x). \]
Since for \( f \in W(\mathbb{R}) \) the sum on the right hand converges to an element \( g_1 \in S(\mathbb{R}) \) with \( \|g_1\|_s \leq \sum_{n \in \mathbb{Z}} \max_{0 \leq x \leq 1} |f(n + x)| \|L_n g\|_s \leq \|f\|_W \|g\|_s \),
\[ f = g_1 \cdot f/g_1 \] is in \( S(\mathbb{R}) \) and satisfies
\[ \|f\|_s \leq \|g_1\|_s \|f/g_1\|_{\infty} \leq \|f\|_W \cdot \|g\|_s \leq C \|f\|_W \quad \text{with } C = \|g\|_s. \]

With some obvious modifications of the proof the above theorem is applicable to the generalization of Wiener's algebra to locally compact Abelian groups \( G \) which have a discrete subgroup \( \Gamma \) such that \( G/\Gamma \) is compact ([3], § 5, example v), in particular to \( G = \mathbb{R}^n \) (with \( \Gamma = \mathbb{Z}^n \), cf. [2], chap. I, § 5, example iii). If we denote these spaces by \( W(G) \) theorem 1 remains true.

In order to give a generalization of \( W(G) \) to arbitrary locally compact groups \( G \) (such that the characterization of theorem 1 extends to this generalization) we introduce a new space \( W^1(G) \). Let \( g \) be any positive function in \( K(G) \). Then \( W^1(G) \) consists of all continuous functions \( f \) on \( G \) which satisfy
\[ |f(x)| \leq \sum_{n \in \mathbb{N}} a_n L_n g(x) \quad \text{for all } x \in G \]
for a suitable sequence \( (y_n)_{n \in \mathbb{N}} \subseteq G \) and a sequence \( a = (a_n)_{n \in \mathbb{N}} \subseteq \mathbb{R} \) with \( \|a\|_1 = \sum |a_n| < \infty \). It is clear that
\[ \|f\| = \inf \{ \|a\|_1, a = (a_n)_{n \in \mathbb{N}} \text{ satisfies } (*) \} \]
defines a norm on \( W^1(G) \). Moreover it is not difficult to see that \( W^1(G) \) does not depend on \( g \); \( g \) can even be replaced by the characteristic function of any open, relatively compact subset of \( G \).

**Theorem 2.** \( W^1(G) \) is a pseudosymmetric Segal algebra on \( G \) which contains \( K(G) \) as a dense subspace and is continuously embedded into \( L^1 \cap C^0(G) \).

**Proof.** Routine computations show that \( (W^1(G), \|\|) \) is a Banach space which is continuously embedded into \( L^1 \cap C^0(G) \) and contains \( K(G) \) as a dense subspace. In particular \( y \rightarrow L_y k \) and \( y \rightarrow R_y k \) is a continuous function from \( G \) to \( W^1(G) \) for all