INTERSECTIONS OF STEINER SYSTEMS $S(3,4,v)$ WITH $v = 5 \cdot 2^n$

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Let $q_v = v(v-1)(v-2)/24$, $I_v = \{0,1,2,\ldots,q_v-14,q_v-12,q_v-8,q_v\}$ for $v \geq 8$, and $J(v) = \{k : \exists$ SQS($v$), $(Q,q_1),(Q,q_2)$ such that $|q_1 \cap q_2| = k\}$. In this paper we show that $I_v - \{q_v-17\} \subseteq J(v)$, for all $v = 5 \cdot 2^n$, $n \geq 2$.

1. INTRODUCTION AND RESULTS

A Steiner quadruple system (SQS) is a pair $(Q,q)$ where $Q$ is a finite set and $q$ is a collection of 4-subsets of $Q$ (called blocks) such that every 3-subset of $Q$ is contained in exactly one block of $q$. The number $|Q|$ is called the order of the SQS $(Q,q)$. In 1960 H. Hanani [8] proved that the spectrum for Steiner quadruple systems consists of all positive integers $v \equiv 2$ or $4 \pmod{5}$. If $(Q,q)$ is an SQS of order $v$ (SQS($v$)), then $|q| = v(v-1)(v-2)/24$. A partial quadruple system (PQS) is a pair $(P,q)$ where $P$ is a finite set and $q$ is a collection of 4-subsets of $P$ (called blocks) such that every 3-subset of $P$ is contained in at most one block of $q$. Two partial quadruple systems $(P,q_1)$ and $(P,q_2)$ are said to be mutually balanced if any given triple of distinct elements of $P$ is contained in a block of $q_1$ if and only if it is contained in a block of $q_2$. Two mutually balanced PQSs are said to be disjoint (DMB PQSs) if they have no block in common. If $(P,q_1)$ and $(P,q_2)$ are any two DMB PQSs then $|q_1| = |q_2|$. An SQS $(R,r)$ is a sub-SQS (or a subsystem) of an SQS

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(Q,q) if R⊆Q and r⊆q. If (R,r) is a subsystem of (Q,q), we will also say that (R,r) is embedded in (Q,q), and that (Q,q) contains (R,r).

Let \( q_v = v(v-1)(v-2)/24 \), \( I_v = \{0,1,2,3,\ldots,q_v-14,q_v-12,q_v-10,q_v-8,q_v\} \) for \( v \geq 8 \), and \( J(v) = \{k : \text{SQS}(v) (Q, q_1), (Q, q_2) \text{ such that } |q_1 \cap q_2| = k\} \). The following question "For a given \( v \equiv 2 \text{ or } 4 \pmod{6} \), for which \( k \leq q_v \) is it possible to construct a pair of SQS(v) having exactly \( k \) blocks in common?" is examined in [4], [5], [6], [7].

Collecting together the results of these papers we have:

**Case \( v = 2^n \)**
1) \( J(4) = \{1\}, \ J(8) = \{0,2,6,14\} = I_8 \); 
2) \( I_v - \{q_v-17\} \subseteq J(v) \), for all \( v = 2^n, n \geq 5 \); \( I_{16} - \{q_{16}-17, q_{16}-18, q_{16}-19\} \subseteq J(16) \);

**Case \( v = 5 \cdot 2^n \)**
3) \( I_v - \{q_v-h : h=17,21,25\} \subseteq J(v) \), for all \( v = 5 \cdot 2^n, n \geq 3 \); 
\( I_{20} - \{q_{20}-h : h=15,17,21,23,25,29,31,39\} \subseteq J(20) \);

and in general
4) \( J(v) \subseteq I_v \), for all \( v \equiv 2 \text{ or } 4 \pmod{6}, v \geq 8 \).

In [9] E.S. Kramer and D.M. Mesner showed that \( J(10) = \{0,2,4,6,8,12,14,30\} \).

The object of this paper is the examination of the block intersection problem for SQS\(s \) having orders \( v = 2^n.5, n \geq 2 \). In particular, we show that:

\( I_v - \{q_v-17\} \subseteq J(v) \), for all \( v = 5 \cdot 2^n, n \geq 2 \).

It is evident the analogy between this result and the results 2). Since the author conjectures that \( q_{16}-18 \) and \( q_{16}-19 \) belong to \( J(16) \) (but this result has not as yet been proved), he thinks that collecting together cases \( v = 2^n \) and \( v = 5 \cdot 2^n \) would give the following result:

\( I_v - \{q_v-17\} \subseteq J(v) \subseteq I_v \).