Infinite Baer Nets*

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In this article, the structure of the collineation groups which fix point-Baer subplanes in vector space nets over skewfields is completely determined. The theory depends on whether there are one, two, or at least three point-Baer subplanes sharing the same parallel classes and a common point.

1 Introduction.

A point-Baer subplane of a projective plane is a subplane such that every point of the plane is incident with a line of the subplane. Similarly, a line-Baer subplane is a subplane such that every line is incident with a point of the subplane. Every finite point-Baer subplane is line-Baer and conversely. However, in the infinite case, the concepts of point-Baer and line-Baer are independent (Barlotti [2]). So, a subplane is Baer if and only if it is both point-Baer and line-Baer. An affine point-Baer subplane is an affine plane which is point-Baer when the plane is considered projectively. A collineation \( \sigma \) of an affine plane which fixes a point-Baer subplane pointwise is said to be a point-Baer perspectivity if and only if the collineation fixes each subplane of a set \( C \) of point-Baer subplanes which form a cover of the points of the affine plane. The collineation \( \sigma \) is a point-Baer elation if and only if \( \text{Fix} \sigma \) is in \( C \). Otherwise, \( \sigma \) is a point-Baer homology. \( C \) is called the center of the collineation, the elements of \( C \) are called the central planes and \( \text{Fix} \sigma \) is the axis.

If a collineation fixes a point-Baer subplane pointwise then, conceivably, it is not a point-Baer perspectivity. However, the structure of point-Baer collineations is essentially completely determined for translation planes. An axial-Baer perspectivity \( \sigma \) is a point-Baer perspectivity such that \( \text{Fix} \sigma \) projectively nontrivially intersects each point-Baer subplane of the center.

The authors have recently provided a general structure theory for point-Baer and line-Baer perspectivities. In particular, the following result is fundamental.

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THEOREM 1.1 (Jha and Johnson [14]). Let $\pi$ be a translation plane and let $\sigma$ be a collineation which fixes a point-Baer subplane pointwise.

Then $\sigma$ is either a point-Baer homology (and hence an axial-Baer homology) or $\sigma$ is an axial-Baer elation and in this case all the planes of the center are proper Baer subplanes. In particular, in all cases, the axis Fix $\sigma$ is a proper Baer subplane of $\pi$ and $\sigma$ has a unique center.

What is not known in general is whether the planes of the center are Baer when $\sigma$ is a point-Baer homology. This can be overcome in certain situations.

THEOREM 1.2 (Jha and Johnson [14]). Let $S$ be a spread in $\text{PG}(3, K)$ for $K$ a skewfield. Let $\pi$ denote the corresponding translation plane.

Let $\sigma$ be a collineation of $\pi$ which fixes pointwise a 2-dimensional $K$-subspace which is not in $S$. Let $\pi_0 = \text{Fix } \sigma$.

1. Then $\sigma$ is in $\text{GL}(4, K)$ and $\pi_0(\sigma - 1)$ is a 2-dimensional $K$-subspace such that $\pi_0$ and $\pi_0(\sigma - 1)$ are line Baer-subplanes that share all parallel classes.
2. $\pi_0$ and $\pi_0(\sigma - 1)$ are point-Baer subplanes so are both Baer subplanes.

Thus, for spreads in $\text{PG}(3, K)$, for $K$ a skewfield, any collineation which fixes a 2-dimensional $K$-subspace which is not in $S$ becomes a Baer collineation with a unique center and axis.

Although the structure theory for a point-Baer collineation $\sigma$ in a translation plane provides information as to the nature of the subplanes of the axis and center of $\sigma$, it says essentially nothing about the collineation groups themselves which fix point-Baer subplanes pointwise. Moreover, the theory says nothing concerning the complete structure of the net defined by the parallel classes of the point-Baer subplane.

The real quaternion plane $\pi$ has its spread in $\text{PG}(3, C)$ where $C$ denotes the field of complex numbers and there is an infinite Baer group acting on $\pi$. So, the existence of any large Baer group which fixes a subplane $\pi_0$ pointwise is not sufficient to ensure that the subplane is left invariant under the kernel of the superplane (is a kernel subplane). On the other hand, for finite translation planes, it has been shown by Foulser ([8]) that a Baer group of order $> 2$ forces the fixed-point subplane to be a kernel subplane. So, a open problem would be to determine Baer groups $B$ such that Fix $B$ is always a kernel subplane.

It turns out that finite planes with spreads in $\text{PG}(3, q)$ that admit large Baer groups (i.e. orders $q$ or $q - 1$) correspond to partial flocks of deficiency one of certain quadric sets ([16]). Furthermore, this theory has been extended by the authors over infinite fields with appropriate changes for the associated groups (see [15]). If one considers a spread in $\text{PG}(3, K)$, for $K$ a skewfield, the study of large point-Baer groups is still possible and a major problem to be resolved is whether the existence of a particular "large" point-Baer group acting on a translation plane with kernel $K$ forces $K$ to be commutative.

In this article, each of the previous problems is considered and resolved. Thus, the theory herein may be regarded as a companion of sorts to the structure theory of point-Baer and line-Baer collineations in affine planes.