Locus Properties of the Neuberg Cubic

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In this paper we explore various locus problems whose solutions involve the Neuberg cubic of the scalene triangle in the plane. We use analytical geometry to show that the Neuberg equation describes the essential part of the locus in each of these problems. In this way we discover new characteristics of the Neuberg cubic that has been at the focus of attention in the recent renaissance of triangle geometry.

1. INTRODUCTION

Let A, B, and C be vertices of a scalene triangle ABC in the plane. In the reference [23], J. Neuberg has considered the following locus problem.

Problem 1. Find the locus of all points P in the plane such that the lines AO\alpha, BO\beta, and CO\gamma are concurrent, where O\alpha, O\beta, and O\gamma denote the circumcentres of the triangles BCP, CAP, and ABP and show that the point of concurrency lies on the line OP where O is the circumcentre of the triangle ABC.

The solution provides the circumcircle of the triangle ABC and the circular cubic N which Neuberg calls the 21-point cubic. The reason for this name lies in the fact that this remarkable curve passes through many important points related to the triangle ABC (for example the following twenty-two points: the vertices, the circumcentre, the orthocentre, the incentre, the excentres, the isogonic points, the isodynamic points, the reflections in opposite sides of the vertices, and the apexes of equilateral triangles constructed on sides).

According to [19] the original 21 points included the three isogonal conjugates of the reflections of the vertices but omitted the four tritangent centres (incentre and three excentres).

The 21-point cubic is known today as the Neuberg cubic of the triangle ABC. It is evident from the extensive list of references on this curve given below that the Neuberg cubic has attracted a lot of attention recently. The present paper is another such contribution.

In this paper we shall use complex numbers to prove that quite an impressive number of other locus determination problems involve the Neuberg cubic. Each of these problems gives
a characteristic of $N$ and sheds a new light on its properties and the way in which it relates to the triangle $ABC$.

Our results are discovered with the help of a computer (PC Pentium 200 MHz, 64 MB RAM) and the software Maple V (version 4).

The paper is organised as follows. After the introduction we describe our notation and give basics on the use of complex numbers in geometry. Then we prove the seven known methods of recognition of the Neuberg cubic. In the remaining sections we present and prove some results of our search for the Neuberg cubic that all give new characterisations of this remarkable curve by various geometric constructions or locus problems. The section titles are chosen to suggest the method of recognition and the theorems in each section are selected so that they have short statements.

Of course, since our results are characterisations of the same curve, in some cases one can show easily that one method follows from the other(s). Observations of this kind and other comments on possible extensions and special cases are included in numerous remarks.

At the end of this paper an alphabetical Glossary of Terms with standard script and Greek script shown in separate sections is provided. This glossary we hope would make the text more readable and save the reader’s time in constantly referring back to earlier pages.

We thank the referee for several suggestions on improving our results and our presentation.

Let us conclude this introduction with the other known locus problems that has the Neuberg cubic as an essential part of the solution. The following two descriptions are also from [23].

For the points $X$ and $Y$, let $|XY|$ denote the distance from $X$ to $Y$. For a point $P$, let $x$, $y$, and $z$ be distances $|AP|$, $|BP|$, and $|CP|$ of vertices of the base triangle $ABC$ to $P$ and let $\ell_a$, $\ell_b$, and $\ell_c$ be lengths $|BC|$, $|CA|$, and $|AB|$ of its sides.

**Problem 2.** Find the locus of the points $P$ in the plane such that the Neuberg matrix

$$
\begin{bmatrix}
1 & \ell_a^2 + x^2 & \ell_a^2 \cdot x^2 \\
1 & \ell_b^2 + y^2 & \ell_b^2 \cdot y^2 \\
1 & \ell_c^2 + z^2 & \ell_c^2 \cdot z^2
\end{bmatrix}
$$

has determinant $D = D(P, ABC) = S (\ell_a^2 \cdot x^2 + \ell_b^2 \cdot y^2 + \ell_c^2 \cdot z^2) = 0$, where $S$ is the cyclic sum.

Another of Neuberg approaches is based on the notion of the power of a point with respect to a circle that we recall now.

Let $P$ be a point and $k$ be a circle in the plane with the centre $S$ and the radius $r$. Then the power $\rho(P, k)$ of the point $P$ with respect to the circle $k$ is the number $|PS|^2 - r^2$. For points $X$ and $Y$ in the plane, let $k(X, Y)$ denote the circle with the centre at $X$ which passes through $Y$.

**Problem 3.** Find the locus of all points $P$ in the plane such that the product of powers of the point $P$ with respect to the circles $k(A, B)$, $k(B, C)$, and $k(C, A)$ is equal to the product of powers of the point $P$ with respect to the circles $k(A, C)$, $k(B, A)$, and $k(C, B)$.