ISOHEDRA WITH NONCONVEX FACES

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An isohedron is a 3-dimensional polyhedron all faces of which are equivalent under symmetries of the polyhedron. Many well known polyhedra are isohedra; among them are the Platonic solids, the polars of Archimedean polyhedra, and a variety of polyhedra important in crystallography. Less well known are isohedra with nonconvex faces. We establish that such polyhedra must be starshaped and hence of genus 0, that their faces must be star-shaped pentagons with one concave vertex, and that they are combinatorially equivalent to either the pentagonal dodecahedron, or to the polar of the snub cube or snub dodecahedron.

We shall use the word polyhedron in an elementary and familiar sense: a polyhedron \( P \) is a bounded three-dimensional solid in ordinary Euclidean space, whose boundary is the union of a finite number of (flat) polygonal regions called faces. The interiors of the faces are assumed to be disjoint, but their boundaries meet in pairs in line segments called edges or in a single point, or not at all. The endpoints of the edges are the vertices of \( P \), and the faces meeting at a vertex \( V \) form a single circuit; hence the boundary of \( P \) is what is usually known as a manifold. The number of faces that meet at \( V \) (which is equal to the number of edges of which \( V \) is an endpoint) is called the valence of \( V \), and the valence of any vertex is at least three.

We note that our interpretation of the word "polyhedron" does not include certain figures which are sometimes called polyhedra in the literature. For example, we exclude the four well-known Kepler-Poinsot polyhedra (see CUNDY & ROLLET [5], or WENNINGER [18] Photos 23 to 26; three are shown in Figure 6 below) since these either have faces with selfintersections, or faces that meet at interior points. The objects called "polyhedra" by topologists often depart in various ways from...
our definition. On the other hand, we do not preclude the possibility that two or more faces of a polyhedron may be coplanar, or that two or more edges of a face may be collinear. An example is shown in Figure 1 of a solid (whose pointset coincides with that of a cube) which we interpret as a polyhedron with 12 faces, 30 edges and 20 vertices; each face has a rectangular shape but since it contains five vertices and five edges it must be regarded as a pentagon. Our definition also does not exclude polyhedra of higher genus, such as the "toroidal" polyhedra shown in Figure 2(a)(b), though, as we shall soon see, these cannot arise in the problem we shall be considering here.

A symmetry of a polyhedron $P$ is any isometry which maps $P$ onto itself. It is easy to verify that every symmetry of $P$, apart from the identity, is either a reflection or a rotation. The set of all symmetries of $P$ forms a group under composition, called the symmetry group $S(P)$ of $P$. Clearly, the centroid $O(P)$ of the vertices of $P$ must be left invariant by all the symmetries of $P$; of course, there may also be other points with this property. In the case where $O(P)$ is the unique invariant point we shall refer to it as the center of $P$. A polyhedron $P$ is isohedral (or, as we shall

![Figure 1](image1.png)

Figure 1. A polyhedron with 12 faces, coplanar in pairs; each face has the shape of a rectangle but is interpreted as a pentagon with two collinear edges. The vertices are indicated by solid dots. This polyhedron has the shape of a cube, but it is combinatorially equivalent to the regular pentagonal dodecahedron.

![Figure 2](image2.png)

Figure 2. Two toroidal polyhedra, one (a) with 9 rectangles and 18 triangles as faces, the other (b) with 36 triangles.