Integration on Supermanifolds and a Generalized Cartan Calculus

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Abstract. A suggestion by Berezin for a method of integration on supermanifolds is given a precise differential geometric meaning by assuming that a supermanifold is the total space of a fibre bundle with connection. The relevant objects for integration are identified as suitable horizontal/vertical projections of hyperforms. The latter are generalizations of differential forms having both covariant and contravariant indices. The exterior calculus of these projected hyperforms is developed, analogously to the Cartan calculus, by introducing appropriate derivations and determining their commutators, respectively anticommutators.

1. Introduction

The concepts of rigid and curved superspace have turned out to be of great importance in current research on supersymmetry and supergravity. As originally introduced by Salam and Strathdee [1] superspace has, besides the coordinates $x^\mu$ ($\mu = 0, \ldots, 3$), which are commuting (even, bosonic), additional anticommuting (odd, fermionic) coordinates $\theta^\alpha (\alpha = 1, \ldots, 4)$. Superfields are functions depending on these variables and encode both bosonic and fermionic fields by means of a Taylor expansion in the odd variables. Integration of superfields with respect to the odd variables is given an operational definition by the Berezin integration rules [2].

Also a (super)-tensor calculus and the notions of (super)-connection, -torsion and -curvature are used frequently in the physics literature [3, 4].

Many authors have investigated how to make these more or less heuristic ideas mathematically rigorous. Rogers [5] introduced the concept of a $D_0 + D_1$ dimensional supermanifold modelled over $B_{L(D_0,D_1)}$, a space obtained from a Grassmann algebra $B_L$. Several modifications of her approach have been proposed [4, 6–8]. The construction of tensor bundles on supermanifolds broadly resembles the procedure for $C^\infty$ manifolds.

What is still lacking is a fully satisfactory theory of integration on supermanifolds mimicking the Berezin integration rules. For $C^\infty$ manifolds the relevant
objects for an integration theory are differential forms. This is not true for supermanifolds. In his last article Berezin [9] pointed out, that in this case one must consider more general tensors with both covariant and contravariant components (which we shall call hyperforms in the sequel). Although he was able to define a consistent supermanifold integral, his definition uses an ad hoc recipe for which we shall give a geometric interpretation. By this means we arrive at a truly geometric, chart-independent integration theory.

In Sect. 2 we make some informal remarks on superspace and supermanifolds. We would like to point out that superspace $B^{(4,4)}_L$ is not yet rigid superspace in physicists' jargon. Rigid superspace is a manifold modelled over $B^{(4,4)}_L$, but with more structure. This is to be compared with Minkowski space, which is a quasi-Riemannian manifold modelled over $\mathbb{R}^4$ with the Poincaré group as isometry group. Similarly a $G^\infty$ function is not the same as a superfield, of which one demands that it transforms according to some representation of the graded Poincaré group. These additional structures are however of no relevance for our arguments.

In Sect. 3 we discuss integration on supermanifolds. The first part of this section is largely a repetition of Berezin's arguments [9] (see also [10–12]), about why one needs hyperforms. Berezin's proposal for a supermanifold integration contains, in our view, a non-geometric ingredient. We overcome this by assuming that the supermanifold is the total space of a bundle with connection. The volume form turns out to be a $P$-hyperform constructed out of suitable pieces of the horizontal and vertical tangent and cotangent spaces of the bundle.

After having identified $P$-hyperforms as the relevant objects for integration on $G^\infty$ supermanifolds (comparable to differential forms for $C^\infty$ manifolds), we improve in Sect. 4 on the previous coordinate-based formulation of Sect. 3. $P$-hyperforms are now obtained from hyperforms by a projection.

Next we aim at an exterior calculus for $P$-hyperforms. We arrive at it in two steps: first in Sect. 5 we develop an exterior calculus for hyperforms, then in Sect. 6 we "project" this onto $P$-hyperforms. The operations are an exterior product $\wedge$, an exterior derivative $d$, contractions with respect to a vector field $i_X$, and derivations obtained from $d$ and $i_X$ by taking suitable commutators or anticommutators. The exterior derivative $d$ is a covariant derivative acting on vector-valued forms. On the space of $P$-hyperforms we can also define a Hodge duality operation.

In our conclusions in Sect. 7 we briefly discuss a possible enrichment of the exterior calculus on $P$-hyperforms by adding further derivations. Finally, since this article is mainly intended to be mathematical, we merely indicate why and how the calculus may be applied to supersymmetric field theories, leaving further details for future articles.

2. Superspace and Supermanifolds

The supermanifolds we are dealing with are modelled over flat superspace $B^{(D_0,D_1)}_L$, the Cartesian product of $D_0$ copies of $B_{L,0}$ and $D_1$ copies of $B_{L,1}$, where $B_{L,0}$ and $B_{L,1}$ are the even, respectively odd, subspaces of a real Grassmann algebra $B_L$ (with $L$ anticommuting generators). Functions from $B^{(D_0,D_1)}_L$ to $B_L$ will be taken to be $G^\infty$,.