A NOTE ON SMALL SIMILARITY GROUPS

Burkhard Alpers*

In this note we consider "small" similarity groups which contain only products of motions and magnifications. Characterizations for this situation are given for regular finite-dimensional metric vector spaces over some fields.

Throughout this note \( \Phi = (V, K, q) \) denotes a regular finite-dimensional metric vector space over a field of characteristic \( \neq 2 \) where \( q \) is the quadratic form. We use the notions of [1] with respect to geometric terminology and that of [8] as far as the algebraic theory of quadratic forms is concerned. Thus, \( D(\Phi) := q(V) \setminus \{0\} \) is the set of nonzero represented elements, and \( G(\Phi) := \{ \lambda \in K^* | \lambda \Phi \cong \Phi \} \) denotes the set of similarity factors of \( \Phi \) where \( K^* := K \setminus \{0\} \). As usual, we denote by \( \Phi \) also its isometry class and the corresponding element in the Witt ring \( W(K) \) of \( K \). If \( 1 \in D(\Phi) \), then \( G(\Phi) \subset D(\Phi) \). Moreover, \( G(\Phi) = G(\alpha \Phi) \forall \alpha \in K^* \), \( G(\Phi) = G(\Phi \bot n \times \langle 1, -1 \rangle) \) for \( n \in \mathbb{N} \), and \( G(\Phi) \cap G(\Psi) \subset G(\Phi \bot \Psi) \) where \( \Psi \) is another regular finite-dimensional metric vector space over \( K \). In [2], we investigated the case \( D(\Phi) = G(\Phi) \) when \( \Phi \) is said to be round. Geometrically, this means that the group of linear similarities acts transitively on the set of non-isotropic vectors (cf. [1]). Here, we want to state some results on the other "extreme" case when \( G(\Phi) = K^{*2} := \{ x^2 | x \in K^* \} \) (note that always \( K^{*2} \subset G(\Phi) \)). From the geometric point of view this means that any linear similarity is a product of a magnification (a mapping of the type \( V \rightarrow V : X \mapsto \lambda X \) where \( \lambda \in K^* \)) and an isometry. In case of \( \dim V \in 2\mathbb{N} + 1 \), we have \( G(\Phi) = K^{*2} \) (cf. [2]). For some fields this turns out to be already a necessary condition, but in general this is wrong.

In the sequel we use frequently the fact that Pfister forms, i.e., forms of the type \( \langle 1, a_1 \rangle \otimes \ldots \otimes \langle 1, a_n \rangle \), are round.

* This note was written during the author's stay as a postdoctoral fellow at the University of Saskatchewan in Saskatoon. The author would like to thank this institution, in particular Prof. Marshall, for the hospitality and the inspiring atmosphere.
1. Let $K$ be a quadratically closed field, i.e., $K = K^2$. Then, we have obviously $G(\Phi) = K^*2$ for any space $\Phi$.

2. Let $K$ be non-formally real and $1 < u(K) \leq 4$ (e.g., finite fields, $p$-adic fields, non-real global fields; for the $u$-invariant cf. [8], p.102). Assume $dim \Phi \in 2\mathbb{N}^*$. If $dim \Phi = 2$, then $\Phi \cong c(1,a)$, hence $G(\Phi) = D((1,a)) \neq K^*2$ since $K$ is not pythagorean (or by Kneser’s lemma, cf. [8],2.16.7). If $dim \Phi \geq 4$, then we may assume $\Phi \cong (1,a,b,c) \perp n \times (1,-1)$ (where $a,b,c \in K^*$), hence $G(\Phi) = G((1,a,b,c))$. In the Witt ring $W(K)$ the equation $(1,a,b,c) = (1,a,b,ab) - ab(1,-abc)$ holds where $(1,a,b,ab)$ is a universal Pfister form since $u(K) \leq 4$. Thus, $D((1,a,b,ab)) \cap D((1,-abc))$ contains a nonsquare $d$, and $d \in G((1,a,b,c))$. Summarizing the above remarks we get for a space $\Phi$ over $K$:

$$G(\Phi) = K^*2 \iff dim \Phi \in 2\mathbb{N} + 1.$$ 

3. Let $K$ be a formally real field with $u$-invariant 2 (e.g., $\mathbb{R}(t)$ or any formally real algebraic extension thereof that is not pythagorean, cf. [3], p. 126). Then, we have again

$$G(\Phi) = K^*2 \iff dim \Phi \in 2\mathbb{N} + 1.$$ 

**Proof.** Let $\Phi$ be an even-dimensional space, $\Phi \cong a_1(1,b_1) \perp \ldots \perp a_n(1,b_n)$. If $s$ is a sum of squares, then $(1,-s) \otimes (1,b_i)$ is torsion, hence hyperbolic. From this we get $s \in D((1,b_i))$ for any nonzero sum of squares $s$ and $i = 1, \ldots, n$. Hence, $G(\Phi)$ contains non-squares.

4. Let $K$ be a formally real global field, and let $dim \Phi \in 2\mathbb{N}^*$. We have $\Phi \cong a_1(1,b_1) \perp \ldots \perp a_n(1,b_n)$. A theorem of Krüskemper ([6], (1.32)) implies that $\bigcap_{j=1}^n D((1,b_j))$ contains infinitely many square classes. Since $G(\Phi) \supset \bigcap_{j=1}^n D((1,b_j))$, we get again

$$G(\Phi) = K^*2 \iff dim \Phi \in 2\mathbb{N} + 1.$$ 

We do not need the strong theorem of Krüskemper that makes use of the Dirichlet-density-theorem. We give the sketch of a proof using the Hasse-Minkowski-theorem as in [4]: First assume $dim \Phi \in 4\mathbb{N}$ and wlog. $1 \in D(\Phi)$. Let $\Phi \cong (1,a_2,\ldots,a_n)$. Then, $\Phi = \langle det \Phi, a_2,\ldots,a_n \rangle + \langle 1, -det \Phi \rangle$ in $W(K)$. By the Hasse-Minkowski-theorem (or by $u(K) = 4$), we have $\lambda(det \Phi, a_2,\ldots,a_n) \cong \langle det \Phi, a_2,\ldots,a_n \rangle$ for any totally positive $\lambda \in K^*$ (cf. [4]). Similar as in the proof of [4](2.2) (using the approximation theorem and the local square theorem), we get that $\langle 1, -det \Phi \rangle$ represents a totally positive non-square $x$. Hence, $x \in G(\Phi) \setminus K^2$. Now, assume $dim \Phi \in 2 + 4\mathbb{N}$, $1 \in D(\Phi)$, and $\Phi \cong (1,a_2,\ldots,a_n)$. Then, $\Phi = \langle 1, -1, -det \Phi, a_2,\ldots,a_n \rangle + \langle 1, det \Phi \rangle$ where $det((1,-1,-det \Phi, a_2,\ldots,a_n)) = 1$. Hence, we can proceed as in the first case.

5. Let $K$ be a formally real pythagorean field which has the strong approximation property (SAP; cf. [7]; e.g., euclidean fields). For $\Phi$ with $dim \Phi \in 2\mathbb{N}^*$, we prove

$$G(\Phi) = K^*2 \iff \text{sgn}_P \Phi \neq 0 \ \forall \text{orderings } P.$$