A REMARK TO "GRAVES TRIADS IN THE GEOMETRY OF THE TRIANGLE" BY A.P.GUINAND

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In his study "Graves triads in the geometry of the triangle" [2], A.P.Guinand has considered the relation between a triangle ABC and certain corresponding triangles whose vertices are on the sides of the triangle ABC. In this study we will prove that the triangle XYZ obtained by Guinand is the polar triangle of the inscribed circle of the reference triangle ABC. Each side of XYZ is the polar of its corresponding vertex with respect to the inscribed circle k of triangle ABC.

Let us denote the inscribed circle of the triangle ABC by k and the outer tangent circles opposite to the vertices A, B and C by k₁, k₂ and k₃ respectively. Let the points of contact of k to the sides a, b and c of the triangle be A₁, B₁ and C₁ respectively. Let us denote the point of contact of k₁ to a by A₂, point of contact of k₂ to b by B₂ and point of contact of k₃ to c by C₂. Let us denote the point of intersection of internal angle bisectors with opposite sides by A₃, B₃ and C₃ and the feet of the altitudes by A₄, B₄ and C₄ (see figure). Hence we get the points of quadruples {A₁,A₂,A₃,A₄}, {B₁,B₂,B₃,B₄} and {C₁,C₂,C₃,C₄} on the sides of ABC. These point quadruples are harmonics and between these the perspectivities

$$A₁A₂A₃A₄ \cong B₁B₂B₃B₄ \cong C₁C₂C₃C₄ \cong A₁A₂A₃A₄$$

exist.

In this study we will prove that there is a polar relation, stated in the following theorem, between the triangle XYZ constituted by the centers of the perspectivities in the relation (1) and the inscribed circle k of the reference triangle ABC.
THEOREM. The triangle XYZ described above is the polar triangle of the reference triangle ABC with respect to its inscribed circle.

Proof: The line joining the points A₁ and C₁ is the polar of B according to the circle k. This line passes through the point Y because of the relation (1). Hence the polar of Y on the polarity of the circle k passes through B. Since the point B is also on the side XZ of the triangle XYZ [2], it is a common point of the polar of Y and the line XZ. The points A₁, A₂ and A₃ on the side a and the points C₁, C₂ and C₃ on the side c of the triangle ABC constitute a Pappus hexagon. The intersection points of opposite sides of this hexagon U=A₂C₃.A₃C₂, V=A₃C₁.A₁C₃ and W=A₁C₂.A₂C₁ are on the Pappus line p. Because of (1) the line p passes through B [1, pp 43-44]. Then the point B is also the common point of the lines p and XZ. On the other hand homogeneous trilinear coordinates of the points A₁, A₃, C₁ and C₃ are A₁(0, \(\cos^2 \frac{C}{2}\), \(\cos^2 \frac{B}{2}\)), A₃(0,1,1), C₁(\(\cos^2 \frac{B}{2}\), \(\cos^2 \frac{A}{2}\),0) and C₃(1,1,0) [2]. According to this the equation of the lines A₃C₁, A₁C₃ and XZ are

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\begin{align*}
x_1 \cos^2 \frac{A}{2} - x_2 \cos^2 \frac{B}{2} + x_3 \cos^2 \frac{B}{2} &= 0 \\
-x_1 \cos^2 \frac{B}{2} + x_2 \cos^2 \frac{B}{2} - x_3 \cos^2 \frac{C}{2} &= 0 \\
x_1(\cos B - \cos A) + x_3(\cos C - \cos B) &= 0
\end{align*}
\]

(2)

respectively, or in another form of writing they are