CHARACTERIZING COMPACT UNIONS OF TWO STARSHAPED SETS IN $R^d$

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Set $S$ in $R^d$ has property $K_2$ if and only if $S$ is a finite union of $d$-polytopes and for every finite set $F$ in bdry $S$ there exist points $c_1, c_2$ (depending on $F$) such that each point of $F$ is clearly visible via $S$ from at least one $c_i$, $i = 1, 2$. The following characterization theorem is established: Let $S \subseteq R^d$, $d \neq 2$. Set $S$ is a compact union of two starshaped sets if and only if there is a sequence $\{S_j\}$ converging to $S$ (relative to the Hausdorff metric) such that each set $S_j$ satisfies property $K_2$. For $S \subseteq R^2$, the sufficiency of the condition above still holds, although the necessity fails.

1. INTRODUCTION

We begin with some definitions from [1]. Let $S$ be a subset of $R^d$. Point $s$ in $S$ is called a point of local convexity of $S$ if and only if there is some neighborhood $N$ of $s$ such that $N \cap S$ is convex. For points $x$ and $y$ in $S$, we say $x$ sees $y$ via $S$ ($x$ is visible from $y$ via $S$) if and only if the segment $[x, y]$ lies in $S$. Similarly, $x$ is clearly visible from $y$ via $S$ if and only if there is some neighborhood $N$ of $x$ such that $y$ sees via $S$ each point of $N \cap S$. Finally, set $S$ is starshaped if and only if there is some point $p$ in $S$ such that $p$ sees via $S$ each point of $S$, and the set of all such points $p$ is called the (convex) kernel of $S$.

A well-known theorem of Krasnosel'skii [3] states that if $S$ is a nonempty compact set in $R^d$, then $S$ is starshaped if and only if every $d + 1$ boundary points of $S$ are visible via $S$ from a common point. In [2], the notion of clear visibility, together with work by Lawrence, Hare, and Kenelly [4], were used to obtain a Krasnosel'skii-type theorem for unions of two starshaped sets in the plane.

A 3-dimensional analogue of this result was established in [1], leading to a characterization theorem for compact unions of two starshaped sets in $R^3$. Here the 3-dimensional characterization is extended to $d$-dimensional sets for all $d \geq 3$.

The following terminology will be used: Conv $S$, aff $S$, cl $S$, int $S$, rel int $S$, bdry $S$, rel bdry $S$, and card $S$ will denote the convex hull, affine hull, closure, interior, relative interior, boundary, relative boundary, and cardinality, respectively, for set $S$. For $x$ in $S$, $A_x$ will represent $\{z : z$ is clearly visible via $S$ from $x\}$. The reader is referred to Valentine [6] and to Lay [5] for a discussion of these concepts.

2. THE RESULTS.

DEFINITION 1. Let $S$ be a finite union of $d$-polytopes $C_1, \cdots, C_n$ in $R^d$. For $F$ a facet of some $C_i$, we say $F$ is a facet of $S$ at point $t$ if and only if for every neighborhood $M$ of $t$, $M \cap F \cap$ bdry $S$ contains a $(d - 1)$-dimensional neighborhood.

DEFINITION 2. Let $S \subseteq R^d$. We say $S$ has property $K_2$ if and only if $S$ is a finite union of $d$-polytopes and for every finite set $F \subseteq$ bdry $S$ there exist points $c_1, c_2$ (depending on $F$) such that each point of $F$ is clearly visible via $S$ from at least one $c_i$, $i = 1, 2$.

Our first result is a $d$-dimensional analogue of [1, Lemma 3].

LEMMA 1. Let $S$ be a compact set in $R^d$, $d \geq 3$, and assume that $S$ is a finite union of $d$-polytopes $C_1, \cdots, C_n$. Let $P$ be a plane in $R^d$, with $B$ a bounded component of $P \cap S$. For $w$ a point of local convexity of cl $B$, $w$ an endpoint of edge $e \subseteq$ rel bdry $B$, there exists a hyperplane $H$ such that the following are true:
1) $H \cap P$ is a line containing $e$.
2) For $N$ an appropriate neighborhood of $w$, $(\text{cl } B) \cap N$ is convex, $B \cap N$ lies in one open halfspace $H_2$ determined by $H$, and $A_w$ lies in the opposite closed halfspace cl $H_1$.

Proof. Clearly cl $B$ is a closed polygonal region with rel bdry $B \subseteq$ bdry $S$. Since every boundary point of $S$ lies in a facet of some $C_i$ set, each point of $e$ lies in a facet of some $C_i$ set. However, there are only finitely many such facets so some of the facets necessarily contain a nondegenerate sequence in $e$ converging to $w$. Let $\mathcal{F}$ denote this collection of facets. Then for $F$ in $\mathcal{F}$, $F$ contains a nondegenerate segment $s_F$ at $w$ along edge