ON 4-TRANSITIVITY IN THE MOUFANG PLANE

Dedicated to Professor H. Karzel on the occasion of his 60th birthday.

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The purpose of this paper is to give a short proof of 4-transitivity in Moufang planes. This proof originated in the observation of the two first name authors that the standard Moufang identities, together with the identity

\[ x^{-1}(y(xz)) = (x^{-1}yx)x, \]

which is asserted in [2, p. 103] to hold in Cayley-Dickson division algebras, can be applied to give a particularly simple algebraic proof of the fact that the collineation group of a Moufang plane is transitive on four-points. Unfortunately, as pointed out by H. Karzel and demonstrated here in Proposition 1, (1) does not hold in Cayley-Dickson algebras. Nevertheless, the algebraic proof of transitivity remains valid after slight modifications and is given here as Theorem 1.

PROPOSITION 1: An alternative division ring \( R \) satisfying (1) is associative.

Proof: We assume that \( R \) is not associative, hence is a Cayley-Dickson algebra over its center \( K \) [3]. Thus \( R \) admits an involution \( x \rightarrow \overline{x} \) for which \( x \overline{x} = n(x)1 \), \( x + \overline{x} = t(x)1 \), for all \( x \in R \) where \( n(x) \in K \), \( t(x) \in K \). Moreover, one knows that \( t(\cdot) \) is a nonzero linear form on \( R \), so \( R \) is spanned by elements \( x \) with \( t(x) \neq 0 \).

For such an \( x \), \( x^{-1} = n(x)^{-1} \overline{x} \) so (1) is equivalent to \( \overline{x}(y(xz)) = (\overline{yx})z \). Now \( \overline{x}(y(xz)) = (t(x)1 - x)(y(xz)) = t(x)y(xz) - x(y(xz)) \) while \( (\overline{yx})z = t(x)(yx)z - (xyz)z \). A standard Moufang identity now yields \( (yx)z = y(xz) \) for all \( y, z \in R \). Since the associative identity is linear in \( x \) and is valid for all elements of a spanning set of \( R \), it holds in \( R \).
We note that a similar result holds also for the identity \((ab)(b^{-1}c) = ac\) which plays an important role in the matrix representability of collineations of a desarguesian plane, namely

**PROPOSITION 2:** An alternative division ring \(R\) satisfying the identity \((ab)(b^{-1}c) = ac\) is associative.

Proof: Suppose \((ab)(b^{-1}c) = ac\). Then \((ab)(b^{-1}c)b = (ac)b\). But \((ab)(b^{-1}c)b = a((b(b^{-1}c))b) = a(cb)\) by a Moufang identity so \(R\) is associative.

Turning now to the main result of the note, we assume that \(R\) is an alternative division ring with center \(K\) of arbitrary characteristic and that \(\Pi(R)\) is the Moufang plane coordinatized by \(R\), [1, p. 219].

**LEMMA 1:** For \(a,b \in R, \alpha, \beta \in K\), the following mappings are collineation of \(\Pi(R)\).

\[
T_{a,b} : (x, y) \rightarrow (x + a, y + b), \quad (x) \rightarrow (x), \quad (\infty) \rightarrow (\infty),
\]

\[
[m, k] \rightarrow [m, -ma + k + b], \quad [k] \rightarrow [k + a], \quad [\infty] \rightarrow [\infty]
\]

\[
R_{a, a \neq 0} : (x, y) \rightarrow (za, aya), \quad (x) \rightarrow (az), \quad (\infty) \rightarrow (\infty),
\]

\[
[m, k] \rightarrow [am, aka], \quad [k] \rightarrow [ka], \quad [\infty] \rightarrow [\infty]
\]

\[
S_{\alpha, \beta, \alpha \neq 0, \beta} : (x, y) \rightarrow (x\beta, y\alpha), \quad (x) \rightarrow (xz\beta^{-1}), \quad (\infty) \rightarrow (\infty)
\]

\[
[m, k] \rightarrow [m\alpha\beta^{-1}, k\alpha], \quad [k] \rightarrow [k\beta], \quad [\infty] \rightarrow [\infty]
\]

\[
F_{a, a \neq 0} : (x, y) \rightarrow (aza, y\alpha), \quad (x) \rightarrow (za^{-1}), \quad (\infty) \rightarrow (\infty)
\]

\[
[m, k] \rightarrow [ma^{-1}, ka], \quad [k] \rightarrow [aka], \quad [\infty] \rightarrow [\infty].
\]

Proof: That these are collineations follows from direct checks of the incidence relation, making use of alternativity and the Moufang identities. For example, in checking \(R_{a}\) we encounter the verification that \(y = mx + k\) implies \(aya = (am)(za) + aka\), a consequence of a Moufang identity.

We can now prove