WEAK STABILITY OF ALMOST REGULAR CONTACT FOLIATIONS

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We prove that on a compact manifold, a contact foliation obtained by a small $C^1$ perturbation of an almost regular contact flow has at least two closed characteristics. This solves the Weinstein conjecture for contact forms which are $C^1$-close to almost regular contact forms.

1. INTRODUCTION

A contact form on a $(2n + 1)$-dimensional manifold $M$ is a 1-form $\alpha$ such that $\alpha \wedge (d\alpha)^n$ is everywhere nonzero. The unique nonsingular vector field $\xi_\alpha$ on $M$ determined by the equations

$$i(\xi_\alpha)\alpha = 1, \quad i(\xi_\alpha)d\alpha = 0,$$

is called the characteristic vector field or the Reeb field of $\alpha$. The flow lines of $\xi_\alpha$ are called the characteristics and the foliation of $M$ by characteristics, the contact foliation.

Thomas [20] introduced the notion of almost regular contact forms: those forms having almost regular characteristic fields, which means that each point in $M$ belongs to a flow box pierced by the flow only a finite number of times. This implies that if $M$ is compact, the orbits are closed 1-dimensional submanifolds, i.e., circles. By a theorem of Wadsley [22], there is a $C^\infty$ circle action with same orbits. If this action is free, then the contact form is said to be regular.

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A compact foliation is a foliation with all leaves compact. For instance, an almost regular contact foliation on a compact manifold is a compact foliation.

A compact foliation is stable if any small perturbation of the foliation has at least one compact leaf. The problem of stability of compact foliations is a deep problem which has been studied by many authors, for instance [7], [8], [17].

The goal of this paper is to prove a weak form of stability of almost regular contact foliations:

**THEOREM.** Let $\mathcal{F}'$ be a contact foliation which is a small $C^1$-perturbation of an almost regular contact foliation $\mathcal{F}$ on a compact manifold. Then $\mathcal{F}'$ has at least two compact leaves.

**REMARKS.**
1. In [2], it was observed that the particular case of this result for regular contact forms can be derived from Ginzburg's work [9].
2. Seifert [18] proved that on $\mathbb{S}^3$, the contact foliation corresponding to Hopf circles is stable. This prompted him to make his famous conjecture (later disproved by Schweitzer [17]).
3. The Weinstein conjecture [23] asserts that the contact flow of a contact compact (simply connected) manifold should have at least one periodic orbit. Our theorem proves this fact for contact forms which are $C^1$-close to almost regular contact forms. For a summary of the Weinstein conjecture and results obtained to date, we refer to [2], [11], [21].
4. A contact form is called R-contact if its characteristic vector field is Killing with respect to some riemannian metric. In [2], it was observed that the Weinstein conjecture is true for R-contact forms as a consequence of the generalization by Molino of the Atiyah-Guillemin-Sternberg momentum map and Molino's theory of riemannian foliations [14], [15]. For R-contact forms, this result can be established by a simplified version of the proof of our theorem. Moreover, in [5], we have found a new, even simpler proof of this fact.

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2. **A LOCAL VARIATIONAL SETTING**

Our strategy is to write down a “generating” function on $M$, which is constant along closed orbits. Therefore its minimum and maximum lie on different closed orbits, and hence there will be at least two closed orbits.