L₁-EMBEDDABILITY OF RECTILINEAR POLYGONS WITH HOLES

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The following result is established. Let $P$ be a rectilinear polygon whose all holes are rectangles. If there are no maximal cut segments of $P$ whose end-points lie on the boundary of different holes then $P$ is $L₁$-embeddable.

In this note we investigate the $L₁$-embeddability of rectilinear polygons with holes endowed with the rectilinear metric. A metric space $(X, d)$ is said to be $L₁$-embeddable if there is a measurable space $(Ω, A)$, a nonnegative measure $μ$ on it and an application $λ$ of $X$ into the set of measurable functions $F$ (i.e. with $∥f∥₁ = ∫_Ω |f(w)|μ(dw) < ∞$) such that

$$d(x, y) = ∥λ(x) - λ(y)∥₁$$

for all $x, y ∈ X$ [1,2,5]. When $Ω$ is a set of cardinality $n$ and $A$ is the collection of subsets of $Ω$ and $μ$ is the cardinality measure, i.e. $μ(A) = |A|$ for $A ⊆ Ω$, then $L₁(Ω, A)$ is called a $n$-dimensional hypercube. A finite metric space $(X, d)$ is hypercube embeddable ($h$-embeddable, for short) if there exist binary vectors $γ₁, \ldots, γₘ ∈ \{0, 1\}^n$ such that $d(x_i, x_j) = ∥γ_i - γ_j∥₁$ for any $x_i, x_j ∈ X$. Problems concerning $L₁$-embeddable and $h$-embeddable metric spaces have a long history, for a survey see [1,2,5].

The interval $I(u, v)$ between two points $u, v$ of a metric space $(X, d)$ consists of all points $x$ between $u$ and $v$, that is,

$$I(u, v) = \{x ∈ X : d(u, v) = d(u, x) + d(x, v)\}.$$ 

A particular instance of a metric space is any connected graph $G = (V, E)$ endowed with the standard shortest-path distance (the distance between vertices $u$ and $v$ of $G$ is the

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number of edges in a shortest path from \( u \) to \( v \). A subset \( S \) of \( X \) is convex if \( I(u, v) \subseteq S \) for any points \( u, v \in S \).

Below we present only those results on \( L_1 \)-embeddable and \( h \)-embeddable spaces and graphs which are needed in the sequel.

**Lemma A** [3,8] A metric space \((X, d)\) is \( L_1 \)-embeddable if and only if \((Y, d_{|Y})\) is \( L_1 \)-embeddable for each finite set of \( X \), where \( d_{|Y} \) denotes the restriction of \( d \) to the set \( Y \).

**Lemma B** If \((X, d)\) is an \( L_1 \)-embeddable metric space then for any points \( u, v \in X \) the interval \( I(u, v) \) is convex.

**Theorem C** [6] Let \( G \) be a bipartite graph endowed with the shortest-path distance. Then \( G \) is \( h \)-embeddable if and only if for any adjacent vertices \( u \) and \( v \) the sets

\[
W(u, v) = \{ z \in V : d(u, z) < d(v, z) \}, W(v, u) = \{ z \in V : d(v, z) < d(u, z) \}
\]

are convex.

This result can be interpreted in the following way. Given two edges \( e = (u, v) \) and \( e' = (u', v') \) of a bipartite graph \( G \), define \( e \theta e' \) if and only if \( u' \in W(u, v) \) and \( v' \in W(v, u) \). The relation \( \theta \) is reflexive, symmetric, but not transitive in general. Djokovic [7] proved that \( G \) is \( h \)-embeddable if and only if \( \theta \) is transitive.

Let \( w = (w_e)_{e \in E} \) be nonnegative weights assigned to the edges of \( G \). The weighting \( w \) is said to be compatible with the relation \( \theta \) if \( w_e = w_{e'} \) whenever \( e \theta e' \).

**Lemma D** [4] Let \( G \) be a bipartite \( h \)-embeddable graph and let \( w \) be a weighting of the edges of \( G \) which is compatible with the relation \( \theta \). Then the resulting metric space is \( L_1 \)-embeddable.

Let \( P \) be a rectilinear polygon in the plane \( R^2 \) (i.e. a polygon having all edges axis-parallel). A rectilinear path \( \pi \) is a polygonal chain consisting of axis-parallel segments lying inside \( P \). The length of the path \( \pi \) in the rectilinear metric is defined as the sum of the length of the segments \( \pi \) consists of. In other words, the length of \( \pi \) is equal to its Euclidean length. For any two points \( u \) and \( v \) in \( P \), the rectilinear distance between \( u \) and \( v \), denoted by \( d(u, v) \), is defined as the length of the minimum length rectilinear path connecting \( u \) and \( v \) [7]. Distance problems on polygons are fundamental in computational geometry and have many applications. A variety of problems such as shortest path queries problems, center and diameter problems, and facility location problems have been studied for various metrics. Rectilinear versions of these problems are motivated by applications in areas, such as VLSI –design, plant and facility layout, urban transportation, wire layout, and robot motion.

Simple rectilinear polygons, being median metric spaces, are \( L_1 \)-embeddable (for the definition of median spaces and related results consult [9]). Any rectilinear polygon \( P \) can be represented as a simple rectilinear polygon inside which lie pairwise disjoint obstacles (holes). Each hole represents the interior of a simple rectilinear polygon. In this paper we address the next question: