THE YANG-ZHANG INEQUALITY

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Yang Lu and Zhang Jinh-Zhong generalized the Neuberg-Pedoe inequality to $\mathbb{R}^n$ by applying Maclaurin’s inequality to a determinant equation. We noticed that a simple proof of an equivalent formula follows from their positive definite matrices alone. The resulting formula also has a simple matrix formulation.

1. INTRODUCTION

Suppose $S(A) = S(a_1, \ldots, a_{n+1})$ is a simplex in $\mathbb{R}^n$ with vertices $a_1, \ldots, a_{n+1}$ and nonzero volume $V(A)$. The vectors $C_i = \overrightarrow{a_{n+1}a_i}$, $i = 1, \ldots, n$ span $S(A)$ at vertex $a_{n+1}$ and the well known formula for $V(A)$ is $V(A) = (1/n!)(\det C^TC) = (1/n!)(\det C)$ where $C_{n \times n}$ has columns $C_i$, $i = 1, \ldots, n$. Since $V(A) \neq 0$, the Gram matrix $C^TC$ is positive definite. Let $a_{ij} = |a_ia_j|$, and $q_{ij} = (a_{i,n+1}^2 + a_{j,n+1}^2 - a_{ij}^2)/2$ and let $Q = (q_{ij})_{n \times n}$. By the law of cosines, $C^TC = (\{C_i, C_j\}) = Q$. Thus $(n!)^2V(A)^2 = \det Q$. Likewise, let $S(B) = S(b_1, \ldots, b_{n+1})$ and $R = (r_{ij})_{n \times n}$ where $r_{ij} = (b_{i,n+1}^2 + b_{j,n+1}^2 - b_{ij}^2)/2$. Let adj $R$ be the adjugate (or adjoint) matrix of $R$ and let $det N_{ij}$ be the minor and $R_{ij} = (-1)^{i+j} det N_{ij}$ the cofactor of $R$ at $r_{ij}$. Finally, a well known application of the geometric-arithmetic mean inequality shows

\begin{equation}
\lambda_1 \ldots \lambda_n = det D \leq \left(\frac{\text{trace}(D)}{n}\right)^n = \left(\frac{\lambda_1 + \cdots + \lambda_n}{n}\right)^n
\end{equation}

if the eigenvalues $\lambda_i$ of $D$ are nonnegative, with equality if and only if all eigenvalues are equal.
2. THE INEQUALITY

THEOREM 1. If $S(A)$ and $S(B)$ are nondegenerate simplices in $\mathbb{R}^n$, then

\begin{equation}
(n!)^2 V(A)^{2/n} V(B)^{2-2/n} \leq \text{trace} \left( Q \ adj R \right) = \sum_{i=1}^{n} \sum_{j=1}^{n} q_{ij} R_{ij}.
\end{equation}

Equality holds if and only if $a_{ij} = t b_{ij}$ for some $t > 0$ and $1 \leq i, j \leq n + 1$.

Proof. Note $Q, R$ and $R^{-1}$ are positive definite symmetric matrices and $Q$ has a positive definite symmetric square root, $Q_0 = Q$ ([4]). Thus $M = Q_0 R^{-1} Q_0$ is also positive definite, so $QR^{-1}$ has positive eigenvalues since $QR^{-1}$ is similar to $M = Q_0^{-1} Q R^{-1} Q_0$. Therefore

\begin{equation}
V(A)^2 V(B)^{-2} = \frac{\det Q}{(n!)^2} \left( \frac{\det R}{(n!)^2} \right)^{-1} = \det QR^{-1} \leq \left( \frac{\text{trace} \left( QR^{-1} \right)}{n} \right)^n,
\end{equation}

so

\[ n V(A)^{2/n} V(B)^{-2/n} \leq \frac{1}{\det R} \text{trace} \left( Q \ adj R \right). \]

The inequality follows by multiplying by $\det R = (n!)^2 V(B)^2 > 0$.

Suppose then that $a_{ij} = t b_{ij}$, where $t > 0$ and $1 \leq i, j \leq n + 1$. Then $Q = t^2 R$, so $QR^{-1} = t^2 I$. Thus all eigenvalues of $QR^{-1}$ equal $t^2$, so equality holds in (3) and thus in (2).

Suppose equality holds in (2) and (3). Then all eigenvalues $\lambda_i$ of $QR^{-1}$ are equal to some $\lambda > 0$. As noticed earlier these are the eigenvalues of $M = Q_0 R^{-1} Q_0$ as well. Since $M$ is real and symmetric there is an orthogonal matrix $P$ such that $P^{-1} M P = \text{diag} (\lambda, \lambda, \ldots, \lambda)$; from $\lambda I = P^{-1} Q_0 R^{-1} Q_0 P$ it follows that $Q = \lambda R$. Comparing the diagonals of $Q$ and $\lambda R$ shows $a_{i,n+1} = \sqrt{\lambda} b_{i,n+1}$, for $1 \leq i \leq n$. This, together with equality of the off diagonal terms, shows $a_{ij} = \sqrt{\lambda} b_{ij}$ for all $i, j$.

3. YANG-ZHANG INEQUALITY

Let $B_{ij}$ be the cofactor of $-b_{ij}^2/2$ in $B = \begin{pmatrix} 0 & 1 & \ldots & 1 \\ 1 & \ddots & \ddots & \vdots \\ \vdots & & \ddots & -b_{ij}^2/2 \\ 1 & & & 1 \end{pmatrix}_{(n+2) \times (n+2)}$.