SPLITTING THE PASCH AXIOM

Victor Pambuccian

On the basis of the theory $\mathcal{E}^-$ of Pasch-free 2-dimensional geometry, Pasch's axiom is shown to be equivalent to the conjunction of the following two axioms: "In any right triangle the hypotenuse is greater than the leg" and "If $\angle AOB$ is right, $B$ lies between $O$ and $C$, and $D$ is the footpoint of the perpendicular from $B$ to $AC$, then the segment $OA$ is greater than the segment $BD". This represents an attempt to split the Pasch axiom with respect to $\mathcal{E}^-$. Only the question whether the second of the above two axioms is really weaker than Pasch's axiom, remains open.

1 INTRODUCTION

In [3] we have shown how the Euclidean parallel postulate can be split into two weaker geometrically meaningful axioms. We have also motivated the operation of splitting axioms in that paper and shall not repeat those arguments here. In this paper, we shall attempt to split the Pasch axiom.

The plane Euclidean geometry of ruler and gauge constructions, considered as a first-order theory in a language $\mathcal{L}$ with two relation symbols $-$ ($\equiv$ (quaternary)) and $B$ (ternary) $-$ will be denoted by $\mathcal{E}$. Its models are Cartesian planes over Pythagorean ordered fields.

Let $F$ be a formally real and Pythagorean field and $\preceq$ an ordering of the additive group of $F$ with $0 \preceq 1$; $\preceq$ will be called a normed semi-ordering of $F$. Let $P = \{x \in F : x \geq 0\}$ be the set of semi-positive elements of $F$, and let $\| \cdot \| : F \times F \to P$ be defined by $\|(x, y)\| = \sqrt{x^2 + y^2}$. Using $\| \cdot \|$, we can define a notion of congruence ($\equiv_F$) and betweenness ($B_F$) by setting $ab \equiv_F cd$ iff $\|a - b\| = \|c - d\|$ and $B_F(abc)$ iff $\|a - b\| + \|b - c\| = \|a - c\|$.  

1Here $v = (v_1, v_2)$ and $v - w = (v_1 - w_1, v_2 - w_2)$.
structure \((F \times F, \equiv_F, B_F)\) will be called a semi-ordered Cartesian plane.

Let \(\mathcal{E}^-\) stand for 2-dimensional Pasch-free Euclidean geometry, the first-order theory, expressed in \(L\), whose models are semi-ordered Cartesian planes. This theory was introduced in L. W. Szczerba \([6]\) and a representation theorem for it was proved in L. W. Szczerba and W. Szmielew \([7]\). All pure congruence-theorems from \(\mathcal{E}\) are in \(\mathcal{E}^-\) as well. The Pasch axiom \((P)\), however, is not in \(\mathcal{E}^-\); by adding \(P\) to \(\mathcal{E}^-\) we get \(\mathcal{E}\).

H. N. Gupta and A. Prestel \([2]\) have considered a weakening of \(P\), which can be equivalently stated as either “The footpoint of the altitude of a right triangle lies between the endpoints of the hypotenuse” or “In every right triangle, the hypotenuse is greater than the legs” \((R)\) or “The triangle inequality”.\(^2\) The equivalence of these statements, provable in \(\mathcal{E}^-\), is shown in [4, Satz 2.3]. It was shown in [2] that the models of \(\mathcal{E}^-\) and \(R\) are quadratically semi-ordered Cartesian planes, that is, the semi-order of the coordinate field \(F\) satisfies the condition
\[
0 < x \to 0 \leq xy^2. \tag{1}
\]

By constructing a quadratically semi-ordered formally real and Pythagorean field which is not ordered, Gupta and Prestel \([2]\) have shown that \(R\) is weaker than \(P\), i.e.
\[
\mathcal{E}^- \not\vdash R \to P.
\]

It is therefore natural to ask for the missing link from \(R\) to \(P\), that is, for a geometrically meaningful statement \(R'\), such that
\[
\begin{align*}
\mathcal{E}^- & \vdash P \leftrightarrow R \land R', \tag{2} \\
\mathcal{E}^- & \not\vdash R' \to P, \tag{3} \\
\mathcal{E}^- & \not\vdash R \lor R'. \tag{4}
\end{align*}
\]

We shall prove that the statement \((R')\): “If \(\angle AOB\) is right, \(B\) lies between \(O\) and \(C\) (with \(B \neq O\) and \(B \neq C\)), and \(D\) is the footpoint of the perpendicular from \(B\) to \(AC\), then the segment \(OA\) is greater than the segment \(BD\)” satisfies \((2)\) and \((4)\). We were unable to either prove or disprove \((3)\).

\(^2\)We shall choose the statement \(R\) as the geometric counterpart of \((1)\), since it requires a smaller number of variables in its formulation in the language \(L\). It can be stated as \((\forall a b b' c) \ o \neq a \land a \neq b \land B(b b') \land ob \equiv ob' \land ab \equiv ab' \land ac \equiv ao \land (B(a c) \lor B(a b)) \to B(a c b)\).