ON THE SMOOTHNESS OF THE CONVEX HULL
IN NEGATIVELY CURVED MANIFOLDS

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In Hadamard manifolds the existence of suitable large convex sets is important for solving the Dirichlet problem at infinity. In this note we prove $C^1$ boundary regularity of the convex hull of any compact set $K$ away from points which lie on geodesics connecting points in $K$.

0. INTRODUCTION

We focus our attention on Hadamard manifolds, that is complete simply connected Riemannian manifolds with non positive sectional curvature. This is mainly to avoid technical difficulties stemming from the possibly complicated topology of the manifold. Besides, the applications would concern these manifolds.

The motivation for studying the smoothness of the convex hull (and more generally the structure of it) is the asymptotic Dirichlet problem for the Laplace operator. This is the following. Let $M^n$ be a simply connected Riemannian manifold with negative sectional curvature. There is a well-known compactification $\overline{M^n} = M^n \cup S^{n-1}(\infty)$, where $S^{n-1}(\infty)$ denotes the ideal boundary or the sphere at infinity. One can now ask: Given a continuous function $f$ on the ideal boundary $S^{n-1}(\infty)$ is there a continuous, harmonic extension to the whole manifold $M^n$?

The answer is affirmative if we have a restriction on the growth of the sectional cur-

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vatures, but it is not known in general. See for example [1], [2], [3], [4]. The crux of the matter seems to be to show that the closed convex hull of a cone has the same intersection with the ideal boundary as the cone itself. For example, it is not known whether the convex hull of a cone can be the whole space.

This is where the study of the structure of convex hulls can become important. We define the convex hull $Con(K)$ of a set $K$ to be the smallest closed convex set containing $K$. We will also need the "first approximation" of the convex hull defined as follows.

$$Con_1(K) = Cl(K) \cup \{ P \in M^3 : \exists \text{ a geodesic } \gamma : (-a_1, a_2) \to M^3 \text{ with } P = \gamma(0) \text{ and } \gamma(-a_1), \gamma(a_2) \in K \},$$

where $Cl(\cdot)$ means the closure. We call a point $P \in Con(K)$ extremal if there is no geodesic arc $\gamma : (-\epsilon, \epsilon) \to C$ with $\gamma(0) = P$.

With variable curvature, one would expect generically for the convex hull of three points to have non-empty interior. It is therefore natural to ask how smooth the boundary of the convex hull is. In connection with this we prove the following:

**THEOREM.** Let $M^3$ be a 3-dimensional complete simply connected Riemannian manifold with non positive sectional curvature. Let $K$ be a compact set and $P \in Con_1(K)$ be a boundary point of $Con(K)$. Then $\partial Con(K)$ is a $C^1$ surface near $P$.

The restriction on the dimension is necessary because the theorem is not true in higher dimension. In dimension two the statement is void because it is easy to see that $\partial Con(K) \subset Con_1(K)$.

There is one obvious question left unanswered: Is $\partial Con(K)$ a $C^2$ surface away from $Con_1(K)$?

1. **PRELIMINARIES**

Throughout the paper the term differentiable means $C^\infty$ if otherwise not specified. $B_t(P)$ will denote the ball with radius $t$ around $P$. We also adopt the convenient notation $O(t)$ and $o(t)$ to denote quantities such that

$$|O(t)| < C \cdot t \quad \text{and} \quad \frac{o(t)}{t} \to 0 \quad \text{as} \quad t \to 0,$$

where $C$ is some positive constant.