HYPERBOLIC UNITALS IN THE HALL PLANES *)

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A new transformation method for incidence structures was introduced in [8], an open problem is to characterize classical incidence structures obtained by transformation of others. In this work we give some sufficient conditions to transform, with the procedure of [8], a unital embedded in a projective plane into another one. As application of this result we construct unitals in the Hall planes by transformation of the hermitian curves and we give necessary and sufficient conditions for the constructed unitals to be projectively equivalent. This allows to find different classes of not projectively equivalent Buekenhout's unitals, [2], and to find the class of unital discovered by Grüning, [4], easily proving its embeddability in the dual of a Hall plane. Finally we prove that the affine unital associated to the unital of [4] is isomorphic to the affine hyperbolic hermitian curve.

1. INTRODUCTION

A unital $\mathcal{U} = (P^*, L^*, I)$ of order $q$ is a $2 - (q^3 + 1, q + 1, 1)$ design, i.e. an incidence structure with point-set $P^*$, line-set $L^*$ and such that $|P^*| = q^3 + 1$, each line is incident with $q + 1$ points, any two different points are incident at unique common line. If $P = (\mathcal{P}, \mathcal{L}, I)$ is a projective plane of order $q^2$ the unital $\mathcal{U}$ is said to be embedded in $P$ whenever: $P^* \subset P$, the incidence relation of $\mathcal{U}$ is induced by the incidence relation of $P$, the lines of $\mathcal{U}$ are traces in $P^*$ of lines of $P$. If $A$ is the affine plane obtained by $P$ deleting a certain line $r$, the set of points of $\mathcal{U}$ not on $r$ is called a unital in $A$; we will denote it by $\mathcal{U}'$ and the points of $\mathcal{U}$ incident with the ideal line $r$ will be called the ideal points of the affine unital $\mathcal{U}'$. Furthermore if the ideal line is a line of $\mathcal{U}$, $\mathcal{U}'$ will be called a hyperbolic unital. Let $q$ be a prime power, it is well known that in the desarguesian plane $PG(2, q^2)$

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the absolute points and non absolute lines of a unitary polarity form an embedded unital of order \( q : H(q) \) which is also called hermitian curve. The hermitian curves of \( PG(2, q^2) \) are projectively equivalent and they can be represented as follows: let \((x, y, z)\) denote the homogeneous coordinates of a point of \( PG(2, q^2) \) and let \([a, b, c]\) denote the line of equation \( ax + by + cz = 0 \), set \( \mathcal{P}^* = \{(x, y, z) \mid x^{q+1} + y^{q+1} + z^{q+1} = 0\} \) and \( \mathcal{L}^* = \{[a, b, c] \mid a^{q+1} + b^{q+1} + c^{q+1} \neq 0\} \), then \( H(q) = (\mathcal{P}^*, \mathcal{L}^*, I) \), \( I \) denoting the incidence relation of \( PG(2, q^2) \). In [2], F. Buekenhout gives a method to construct unitals which embed in non-desarguesian planes. Precisely he constructed unitals in every affine translation plane of order \( q^2 \) with a kernel of order \( q \) : if \( H(q) \) is a classical unital in the Desarguesian affine plane \( A = AG(2, q^2) \), then the set of points of \( H(q) \) forms a unital in each affine translation plane \( \tilde{A} \) such that:

1. The points of \( \tilde{A} \) are the points of \( A \);
2. Translations of \( A \) are translations of \( \tilde{A} \);
3. Lines of \( A \) through points of \( H(q) \) on the ideal line are lines in \( \tilde{A} \);
4. \( \tilde{A} \) has \( GF(q) \) in its kernel. (For the notion of kernel see [3]).

For short we will call these unitals the Buekenhout’s unitals.

Later K. Grünig [4] constructed a unital embedded in the Hall plane of order \( q^2 \) and in its dual plane also. His construction is similar to that used by Buekenhout, in fact he also used the method of André [1] to represent translation planes in a four-dimensional projective space. Precisely let \( A = AG(2, q^2) \), then \( A \) can be represented in a four-dimensional projective space and the ideal points of \( A \) are the lines of a regular spread \( S \). If \( H' \) is an hyperbolic hermitian curve of \( A \) whose ideal points correspond to the lines of a regulus \( R \) in \( S \), then, denoting \( \hat{R} \) an opposite regulus of \( R \), the points of \( H' \) together with the lines of \( \hat{R} \) constitute a unital in the Hall plane obtained from \( A \) replacing \( R \) by \( \hat{R} \).

We will call this unital a unital of Grünig.

In this work, applying a new method, we construct unitals in the Hall plane of order \( q^2 \) reobtaining different classes of not projectively equivalent Buekenhout’s unitals and the class of unitals found by Grünig; furthermore we give a new proof of the embeddability of this unital in the dual of the Hall plane.

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2. CONSTRUCTION OF HALL PLANES AND EMBEDDED UNITALS

In [8] a method to transform incidence structures was defined. We recall it applied to the case of a projective plane of finite order.

Let \( P = (\mathcal{P}, \mathcal{R}, I) \) be a projective plane of order \( q^2 \), let \( \mathcal{F} \subset \mathcal{R} \) be a family of lines and \( f \) a permutation on the point-set \( \mathcal{P} \). For every couple of points \( P_1, P_2 \) let \( < P_1 P_2 > \) denote the line through them.

We say that \( P \) is transformable and \( \{\mathcal{F}, f\} \) is a transformation system for \( P \) if for every couple of distinct points \( P, Q \in \mathcal{P} \), the following condition is satisfied:

\[
< PQ > \in \mathcal{F} \iff < f(P)f(Q) > \in \mathcal{F}
\]

It is possible to define a new point-block incidence relation \( J \) as follows: