ON THE ALGEBRAIC REPRESENTATION OF PROJECTIVELY EMBEDDABLE AFFINE GEOMETRIES

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The main result of this article is an application of [1] and [2] which yields that an at least 2-dimensional affine geometry is module-induced if and only if it is projectively embeddable into an Arguesian projective lattice geometry.

1 INTRODUCTION

A general concept of affine geometry which includes the congruence class geometry of modules has been introduced in [7]. In this article we first show how to each projective lattice geometry as established in [4] with a fixed hyperplane there is naturally associated an affine geometry; isomorphic copies of the latter will be called projectively embeddable affine geometries (cf. 2.8). Obviously, each module-induced affine geometry is projectively embeddable.

Our main question concerns the converse: when is a projectively embeddable affine geometry module-induced? Fortunately this difficult problem was already implicitly treated in [1] and later in [2]. A main lack of the afore was that the authors could not exploit the full results of their investigations, since an appropriate concept of affine geometry was not developed then.

Now (within the conceptual frame of [7] we are able to prove, as a nice application of the results of the authors mentioned before, that every at least 2-dimensional affine geometry is module-induced provided it is projectively embeddable into an Arguesian projective lattice geometry (cf. 2.10). This generalizes the classical representation for classical affine geometries by vector spaces. As an open problem we mention that it is not known yet whether every projectively embeddable affine geometry of dimension at least 3 is module-induced.
2 MAIN PART

From [7, p. 97] we introduce an affine geometry as a triple \((V, \parallel, \%\)) satisfying the following axioms:

(A) **Axiom of incidence**: \(V\) is an atomistic algebraic lattice (with order relation \(\leq\)).\footnote{i.e. \(V\) is a complete lattice which is compactly atomistic in the sense of [6].}

**Comment**: The join (i.e. the least upper bound) or the meet (i.e. the greatest lower bound) of any two elements \(x, y \in V\) will be denoted by \(x \lor y\) or \(x \land y\), respectively; the least element of \(V\) is abbreviated by 0, and \(P\) will be the set of atoms of \(V\).

(B) **Axiom of parallelity**: \(\parallel\) is a parallelism on \(V\), i.e. \(\parallel\) is an equivalence relation on \(V \setminus \{0\}\) satisfying:

(B1) *Euclid’s parallel postulate*: For \(x \in V \setminus \{0\}\) and \(p \in P\) there always exists a unique element \(y\) of \(V\) with \(p \leq y\) and \(x \parallel y\); abbreviation: \(\pi(p | x) := y\).

(B2) *Monotony property*: For \(x, y \in V \setminus \{0\}\) with \(x \leq y\) and \(p \in P\) it always follows \(\pi(p | x) \leq \pi(p | y)\).

**Comment**: If \(x \parallel y\) holds we say that \(x\) and \(y\) are parallel; \(x\) is lower parallel to \(y\), abbreviated by \(x \leq \parallel y\), whenever \(x \leq t \parallel y\) holds for some \(t \in V\); lower parallelity defines a quasi-order on \(V \setminus \{0\}\).

(C) **Axioms of reducibility**:

(C1) For all \(a, b, c \in P\) there exists \(d \in P\) with \(a \lor b \parallel c \land d\).

(C2) For all \(a, b, c \in P\) and \(x, y \in L\) with \(a \leq x \parallel y \leq x \lor b\) it follows \((a \lor b) \land y \neq 0\).

(D) **Axioms of independence**: \(\%\) is an affine independence relation on the atoms of \(V\), i.e. \(\%\) is an antireflexive, symmetric binary relation on \(P\) satisfying:

(D1) For all \(a, b, c \in P\) with \(a \% b\) there exists \(d \in P\) with \(c \% d\) and \(a \lor b \parallel c \land d\).

(D2) For all \(a, b, c \in P\) and \(x \in V\) with \(a \% b\) and \((a \lor b) \land x = a\) it follows \(a' \% b\) and \((a' \lor b) \land x = a'\) for all \(a' \in P\) with \(a' \leq x\).

**Comment**: If \(a \% b\) holds we say that \(a\) and \(b\) are independent (or distant); \(\%\) is crucial for the notion of basis and dimension of an affine geometry (as we will see below).

**REMARK 2.1** Axioms (A), (B) together with (C) imply that \((V, \parallel)\) is reconstructable from an associated point-line structure which has been called affine space in [7].

**EXAMPLE 2.2** To each module \(RM\) there is naturally associated its affine geometry \(AG(RM) := (V, \parallel, \%)\) where