ON STRICT STRONG CONSTRUCTIBILITY WITH A COMPASS ALONE

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We show that every point in the plane which can be constructed by a compass and a ruler, given a set $S$ of points, can be constructed using a compass alone so that the following restriction is met. Let $O$ and $K$ be two arbitrarily chosen distinct points of $S$. Then every point is obtained as a proper intersection of two circles that are either completely symmetrical with respect to the line $OK$ or have both their centers on this line.

In [1] we have shown that every point in the plane which can be constructed by a compass and a ruler, given a set $S$ of points, can be constructed using a compass alone so that the centers of all the circles used are on a particular line $OK$, where $O$ and $K$ are two arbitrarily chosen distinct points of $S$. This was a strengthening of a famous theorem of Mascheroni and Mohr. There was, however, a serious drawback in our construction: points on the line $OK$ itself were (necessarily) obtained only as the tangent points of two circles and not as proper intersection points. The original proofs of Mascheroni and Mohr, in contrast, took special care to avoid using tangent points.\(^{(1)}\) In this paper we remedy this shortcoming. For this we shall somewhat relax, of course, the restriction above. Nevertheless, the needed relaxation turned out to be minimal: points outside $OK$ are still obtained as the (proper!) intersection points of two circles with centers on $OK$, but the points of $OK$ itself are obtained as the intersection points of two circles which are completely symmetrical with respect to $OK$.

In the definition below $S$ is a set of points in the plane, $O$ and $K$ are two distinct points of $S$.

\(^{(1)}\) This point was called to our attention by the referee of [1]. We take the opportunity here to thank him.
DEFINITION. 1) A construction with a compass alone of a point $B$ from $S$ is a sequence $A_1, \ldots, A_n = B$ of points such that for each $1 \leq i \leq n$ either $A_i \in S$ or there exist in the sequence points $A_{i_1}, A_{i_2}, A_{i_3}, A_{i_4}, A_{i_5}, A_{i_6}$ such that $i_j < i$ ($1 \leq j \leq 6$), $A_{i_2} \neq A_{i_3}$, $A_{i_5} \neq A_{i_6}$, and $A_i$ is an intersection point of $A_{i_1}(A_{i_2}A_{i_3})$ and $A_{i_4}(A_{i_5}A_{i_6})$. (2)

2) Two circles are completely symmetrical with respect to a line $\ell$ iff their centers are symmetrical with respect to $\ell$ and their radii are equal.

3) We call a construction from $S$ with a compass alone permissible (relative to $O$ and $K$) if any point it uses which is not in $S$ (including the final one) is obtained as a proper intersection of two circles which are either completely symmetrical with respect to $OK$ or have both their centers on $OK$.

4) We shall call a point $C$-constructible (from $S$ relative to $O$ and $K$) if it can be obtained from $S$ by a permissible construction (relative to $O$ and $K$).

THEOREM. Every point of the plane that can be constructed from $S$ using a ruler and a compass is $C$-constructible from $S$ relative to $O$ and $K$ where $O$ and $K$ are arbitrarily chosen two distinct points of $S$.

Proof. The proof closely follows that given in [1], though some of the constructions there need to be changed. We leave to the reader the task of checking that every construction we use below is permissible. Again we employ a Cartesian coordinate system in which $O = (0,0), K = (1,0)$.

Fact 1. Suppose $A$, $B$, $C$ are on the $X$-axis, and $AB = BC$. Then if $A$ and $B$ are $C$-constructible then so is $C$.

Proof. Let $C_1$ and $C_2$ be the intersection points of $A(AB)$ and $B(AB)$. Then $C_1C_2 = \sqrt{3}AB$, and $C$ is one of the two intersection points of $C_1C_2$.

Fact 3. (Corollary): if $(x, 0)$ is $C$-constructible, then so is $(nx, 0)$ for every integer $n$.

Fact 4. If $(x, 0)$ is $C$-constructible so is $(0, x)$.

Proof. Exactly like in Lemma 5 of [1]: By Fact 2 $(-x, 0)$ is $C$-constructible. Let $A = (x, 0), B = (-x, 0).$ Then $AB = 2x$. Now $A(2x)$ and $B(2x)$ intersect at $(0,\sqrt{3}x)$, $A(\sqrt{3}x)$, $B(\sqrt{3}x)$ intersect at $(0,\sqrt{2}x)$ and $A(\sqrt{2}x)$, $B(\sqrt{2}x)$ intersect at $(0, x)$.

$(2)$ $A(BC)$ is the circle with center at $A$ and radius $BC$. 