THE DISCRETE VERSION OF A GEOMETRIC DUALITY THEOREM

Dedicated to Professor Srinivas Ramanujan on his birth centenary

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It was shown by Rhodes [1] that a theorem about subsets in the plane specified by the Euclidean metric generalizes to an interesting duality between the absolute and the maximum metrics in the real plane. In this paper the discrete version of this duality is shown to hold between the cityblock (absolute) and the chessboard (maximum) metrics in the quantized space. The characterization of the 'bisector' and the 'near-bisector' under the above metrics is obtained as a by-product.

1. INTRODUCTION

Let $R$ be the set of real numbers and $R^+$ be the set of non-negative real numbers. Hence $R^2 = \{(x_1, x_2) : x_1, x_2 \in R\}$ is the real plane. If $d : R^2 \times R^2 \to R^+$ is a metric (i.e., $d$ is total, positive definite, symmetric and triangular) on $R^2$ then we can define the following quantities in terms of $d$.

(A) Set of Disk Points: $D(x, r) = \{(u : u \in R^2, d(x, u) \leq r \text{ and } r \in R^+\}$ where $x \in R^2$ is the centre and $r$ is the radius.

(B) Set of Between Points: $B(x, y) = \{(u : u \in R^2, d(x, u) + d(u, y) = d(x, y)) ; x, y \in R^2\}$. From the triangularity of $d$, $B$ is clearly the set of points lying on some minimal path from $x$ to $y$ as defined by the metric $d$.

(C) Set of Equidistant (Bisector) Points: $E(x, y) = \{(u : u \in R^2, d(x, u) = d(y, u)) ; x, y \in R^2\}$. 
If \(d\) is a norm, that is, \(d(x,y) = d(Q, x-y)\) where \(Q = (0,0)\) is the origin then clearly there exists a bijection between \(E(x,y)\) & \(E(Q, x-y)\) and \(B(x,y)\) & \(B(Q, x-y)\). Hence without loss of generality we may use \(y = 0\) and write \(E(x,0)\), \(B(x,0)\) and \(d(x,0)\) simply as \(E(x)\), \(B(x)\) and \(d(x)\) respectively.

In particular, if \(d = e\), the Euclidean distance \(e(x,y) = [(x_1-y_1)^2 + (x_2-y_2)^2]^{1/2}\) then \((A)\) is the circle of radius \(r\) centred at \(x\), \((B)\) is the straight segment joining \(x\) and \(y\), and \((C)\) is the perpendicular bisector of \((B)\). Immediately we can recall a theorem about subsets in the plane specified by the Euclidean metric to relate these quantities as:

\[
B(x,y) = \bigcap_{u \in E(x,y)} D(u, e(u,x)) \tag{1}
\]

that is, the infinite intersection (common part) of all the circular discs whose centres are on \(E(x,y)\) and whose boundaries pass through \(x\) and \(y\) is the segment \(B(x,y)\).

It is an interesting exercise in any non-Euclidean metric geometry defined on \(R^2\), to explore the relation between the left-hand and right-hand sides of equation (1). The simplest such example concerns the extreme \(L_p\)-metrics, namely the absolute metric \(d_A = L_1\) and the maximum metric \(d_M = L_\infty\), where \(d_A(x,y) = |x_1-y_1| + |x_2-y_2|\), \(d_M(x,y) = \max(|x_1-y_1|, |x_2-y_2|); x,y \in R^2\). Rhodes [1] has pointed out an exciting duality in (1) in the case of these metrics. Using appropriate suffixes for the quantities defined in \((A)\), \((B)\) & \((C)\), the following duality theorem has been proved in [1].

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d_A - d_M DUALITY THEOREM
\]

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PART 1: \quad B_A(x,y) = \bigcap_{u \in E_M(x,y)} D_M(u, d_M(u,x)) \tag{2}
\]

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PART 2: \quad B_M(x,y) = \bigcap_{u \in E_A(x,y)} D_A(u, d_A(u,x))
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