HEXAGONAL-SURFACE-WEBS FORMED BY n-PENCILS OF SPHERES BELONGING TO THE SAME BUNDLE 1)

Hakki Ismail ERDOĞAN

In this work, it is shown that n pencils of spheres which belong to the same bundle form a hexagonal-surface-web. Firstly, 4 pencils of spheres orthogonal to the same sphere are taken into consideration. Later, by means of a suitable transformation, the equations of these 4 pencils of spheres are written in their simplest form and the equation of the surface web is obtained. Then, it is concluded that any 4-pencils of spheres belonging to the same bundle form a hexagonal-surface-web. From this we conclude that a surface n-web which is formed by n pencils of spheres belonging to the same bundle is a hexagonal web.

1. PRELIMINARIES

1.1 SURFACE WEBS

By a surface web in a region H of 3-dimensional space, with cartesian co-ordinates x, y, z, is meant the configuration formed by four families of surfaces

\[ U_i(x, y, z) = U_i = \text{constant}, \quad (i = 0, 1, 2, 3) \]

where one and only one surface from each family passes through every point \((x, y, z)\) of the region H and the four functional determinants

\[ \Delta(U_i, U_j, U_k) \quad \frac{\Delta(U_i, U_j, U_k)}{\Delta(x, y, z)} \quad (i, j, k = 0, 1, 2, 3; \quad i \neq j \neq k) \]

are not zero.

1) n pencils of spheres are said to belong to the same bundle, if all the spheres cut a fixed sphere orthogonally.
In this case, any three families of these surfaces cut a web of curves on the surfaces of the fourth family. In this way four webs of curves are obtained which we shall denote by \( \mathcal{H}, \mathcal{M}, \mathcal{M}_2, \mathcal{M}_3 \). It is well known that if any three of these webs are hexagonal, then the fourth one is also hexagonal [1, p. 30].

1.2 HEXAGONAL-SURFACE-WEB

A surface web is said to be a hexagonal-surface-web, if on each surface of one family the surfaces of the remaining three families cut a hexagonal web.

2. FORMULATION OF THE PROBLEM

To start with, let us take 4-pencils of spheres belonging to the same bundle (i.e., all the spheres cut a fixed sphere orthogonally). If we take inversion with respect to any point of the fixed sphere, the fixed sphere transforms to a plane (e.g., Oxy-plane) while the 4-pencils of spheres transform to 4-pencils of spheres the centers of which are on that plane.

As is known, a pencil of spheres can be given either by

\[
K_1 + \lambda_1 K_2 = 0, \tag{2.1}
\]

where \( K_1 = 0 \) and \( K_2 = 0 \) are two given spheres, or by using the radical plane of the pencil and one of the spheres \( K_1 = 0 \) and \( K_2 = 0 \).

In the latter case the pencils of spheres are given by

\[
K_1 + \lambda (K_1 - K_2) = 0, \tag{2.2}
\]

which is more suitable for algebraic operations.

It is obvious that the radical planes of all pairs of spheres belonging to the same pencils are the same.

Therefore, the equations of 4-pencils of spheres which are inversions of 4-pencils of spheres belonging to the same bundle, can be written as

\[
K_i := x^2 + y^2 + z^2 + a_i x + b_i y + c_i + U_i (a_i x + b_i y + y_i) = 0, \quad (i = 1, 2, 3, 4) \tag{2.3}
\]

Where \( U_i \)'s are pencil parameters. Consequently, elimination of \( x, y, z \) from the equations (2.3) leads to the web equation of the surface web. Namely,