MULTIPlicITIES OF SINGULAR POINTS ON ARCS AND CURVES OF CYCLIC ORDER FOUR

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It is known that a strongly differentiable ([3], 3.1) curve \( C_4 \) of cyclic order four in the real conformal plane contains at most (in fact exactly) four singular points ([4], 3.6 and [1], 4.1.4.3). Assuming no differentiability conditions the best bound obtained was eleven ([4] and [1], 4.1.3) and then reduced to the best possible bound; namely four ([5], 4.1). In this paper multiplicities are introduced for singular points on such arcs and curves. It is shown that the sum of the multiplicities of all singular points on arcs and curves of cyclic order four is indeed at most four; cf. 5.5.

1. PRELIMINARIES.

1.1 A point \( p \) on an arc \( A \) in the real conformal plane is said to be (conformally) differentiable [2] if it satisfies two conditions:

I For every point \( R \neq p \) and for every sequence of points \( s \rightarrow p \) on \( A \) there exists a circle \( C_0 \) such that \( C(s,p,R) + C_0 \). \( C_0 \) is called the tangent circle of \( A \) at \( p \) through \( R \) and is denoted \( C(p^2,R) \).

II If \( s \rightarrow p \) on \( A \) there exists a circle \( C(p^3) \) such that \( C(p^2,s) + C(p^3) \). \( C(p^3) \) is called the osculating circle of \( A \) at \( p \). \( C(p^3) \) may be the point circle \( p \). For simplicity \( C(p^3) \) will be abbreviated to \( C(p) \).

A point \( p \) on \( A \) is said to be strongly differentiable if the following are satisfied:

CI Let \( R \neq p, Q \rightarrow R \). If two distinct points \( u, v \rightarrow p \) on \( A \), then \( C(u,v,Q) \) converges.
CII. $C(t,u,v)$ converges if the three distinct points $t,u,v+p$ on $A$.

Some results concerning differentiability are [3]:

(i) The set of all tangent circles all touch each other at $p$

(the set of all tangent circles is a pencil of the second kind with the fundamental point $p$).

(ii) Nontangent circles through $p$ all intersect or all support.

(iii) The nonosculating tangent circles through $p$ all intersect or all support. If $C(p) \neq p$, all of them support.

(iv) Strong differentiability implies ordinary differentiability.

(v) Strong differentiability implies that the osculating circle varies continuously with $p$.

1.2 A differentiable interior point $p$ of an arc $A$ has the characteristic [3] $(a_0,a_1,a_2)$ if $C(p) \neq p$ or $(a_0,a_1,a_2)_0$ if $C(p) = p$ where $(a_2 = \infty$ will not be considered here):

(i) $a_0,a_1,a_2$ are equal to 1 or 2.

(ii) $a_0[a_0 + a_1]$ is even or odd accordingly as the nontangent circles [the nonosculating tangent circles] at $p$ support or intersect.

(iii) $a_0 + a_1 + a_2$ is even if $C(p)$ supports, odd if $C(p)$ intersects.

1.3 The cyclic order of an arc $A$ is the maximum number of points in common with any circle. The order of a point $p$ is the minimum of the orders of all neighbourhoods of on $A$.

A point of (minimal) order three is called an ordinary point, a point of order greater than three a singular point, and a point of support of $A$ with respect to $C(p)$ a vertex.

This paper involves only (simple) arcs $A_4$ of cyclic order four in the real conformal plane. With regard to such arcs:

(i) there are at most four singular points [5],

(ii) points with the characteristic $(1,1,1)$ are exactly the ordinary points while the singular points are vertices and have the characteristic $(1,1,2), (1,1,2)_0, (1,2,1)_0, (2,1,1)_0$ [3],

(iii) points with the characteristic $(1,1,1), (1,1,2)$ or $(1,1,2)_0$ are strongly differentiable [3].