An M-space is a metric space $X$ with distance function $d(\cdot, \cdot)$ having the property that for each pair of points $p, q \in X$ with distance $d(p, q) = \lambda$, and for each real number $\alpha \in [0, \lambda]$, there is a unique point $r_{\alpha} \in X$ such that $d(p, r_{\alpha}) = \alpha$ and $d(r_{\alpha}, q) = (\lambda - \alpha)$. The present paper extends to M-spaces previous results relating strict convexity, monotonicity of the distance function, and Chebyshev sets in other classes of metric spaces.

In [8] Khalil defines a concept of M-space, and proves results relating proximinality of sets to the structure of the space. The present paper extends to M-spaces previous results relating strict convexity, monotonicity of the distance function, and Chebyshev sets in other classes of metric spaces.

Following Khalil, we call a metric space $X$ with distance function $d(\cdot, \cdot)$ an M-space provided for every two points $p, q \in X$ with distance $d(p, q) = \lambda$, and every real number $\alpha \in [0, \lambda]$, there is a unique point $r_{\alpha} \in X$ such that $d(p, r_{\alpha}) = \alpha$ and $d(r_{\alpha}, q) = (\lambda - \alpha)$. It is noted that these M-spaces have unique betweenpoints, and hence unique metric segments, but do not necessarily have unique metric lines (or metric lines at all). Furthermore, no explicit assumption of completeness is made. Hence, while metric segments are complete, the property is not necessarily valid for arbitrary point sets in the space. M-spaces might be observed to bear the same relationship to complete convex metric spaces as do normed linear spaces to Banach spaces.

In the remainder of the paper we will denote by $pq$ the distance $d(p, q)$ between points $p$ and $q$. A closed subset $G$ of $X$ is proximinal provided for each $p \in X$ there is at least one point $f$ in $G$, called a foot of $p$ on $G$, such that $pf = \inf\{pq : q \in G\}$. If this point $f$ is unique for each $p \in X$, the set $G$ is called a Chebyshev set. Recall that a metric space $X$ is strictly
convex (or has strictly convex spheres) provided if \( p, q, r \in X \) with \( pq = pr \), then for every point \( s \) metrically between \( q \) and \( r \), \( ps < pq \). An M-space \( X \) is externally convex provided for each two distinct points \( p, q \in X \), there exists a point \( r \in X \), \( r \neq q \), such that \( pq + qr = pr \) (denoted \( pqr \)). The space \( X \) is strongly externally convex provided for all distinct points \( p, q \in X \) such that \( pq = \lambda \), and for all \( \kappa > \lambda \), there exists a unique point \( r \in X \) such that \( pq + qr = pr = \kappa \).

1. EQUIVALENTS OF STRICT CONVEXITY

Previous results relating strict convexity, monotonicity of the distance function, and Chebyshev sets obtained by Busemann [4], Freese and Murphy [6], and Freese and Andalafte [5], can be generalized in two ways, first by deriving the results in M-spaces, and then by observing that these equivalences remain valid in complete, convex metric spaces. The space \( X \) has the monotone property provided the distance \( px \) from a point \( p \) to a point \( x \) on a metric line, metric ray, or closed metric segment \( L \) is (strictly) monotone increasing as \( x \) recedes in either direction from any foot of \( p \) on \( L \). A space has the isosceles weak strict ptolemaic property provided each non linear quadruple \( p, q, r, s \) for which \( qr, pq = ps \), and \( qr = rs \) holds, satisfies the strict ptolemaic inequality. The space has the isosceles feeble strict ptolemaic property provided each non linear quadruple \( p, q, r, s \) for which \( qrs, pq = ps \), and \( qr = rs \) holds, satisfies the strict ptolemaic inequality. The first result is the following theorem.

**THEOREM 1.1:** In an M–space \( X \) the following are equivalent:

i) Proximinal convex subsets of \( X \) are Chebyshev sets,

ii) \( X \) has the monotone property,

iii) \( X \) is a strictly convex space,

iv) \( X \) has the isosceles feeble strict ptolemaic property, and

v) \( X \) has the isosceles weak strict ptolemaic property.

**Proof:** The proof that i) implies ii) is almost exactly as in [2] (Lemma 1.1). Suppose i) and suppose the monotone property is not satisfied. Then there exists a metric line, closed metric ray, or closed metric segment \( G \), a point \( p \in X - G \), a foot \( f \) of \( p \) on \( G \), and points \( q, r \in G \) such that \( fqr \) and \( pq = pr \). Since the foot of a point on \( G \) is unique, the foot \( f(t) \) of a point \( t \) on \( G \) is a continuous function of \( t \) on any metric segment joining \( p \) and \( r \), so there is a point \( s \) metrically between \( p \) and \( r \) whose foot on \( G \) is \( q \). Now \( sr = pr - ps = \)