ABOUT FULL OR INJECTIVE LINEATIONS

To the memory of Iulian Popovici

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Some geometric properties of full lineations are given and used to extend an affine full lineation to the projective envelope of its domain. A criterion for injective lineations to be full and surjective is given; in connection with this some bad injective lineations are constructed and studied.

1. INTRODUCTION

The sources of this paper are the characterization of lineations between projective or affine planes (resp. spaces) from [8], [4], [5] (resp. [7]) and the characterization of injective lineations between real projective spaces obtained in [2]. This paper is an improved form of [3], I. We heartily thank Prof. F. Radó for the suggestion to consider non-injective lineations and for an improvement in 3.4, which led to the present form of the paper. In the proof of 3.4, the argument proposed by F. Radó to show that $F$ is full on $V_j^v_j$ eliminates the restriction char $K \neq 2$ (compare with [3], I, 3.6). We also heartily thank the referee for his kind criticism and for the generalization of 5.4 and 5.5 ii); we reproduce his enunciations and proofs of 5.4 and 5.5, compare with [3], I, 5.4, 5.5 ii). We are indebted to our late friend Dr. Iulian Popovici from whom we acquired our geometric skill.

Let $m \geq 2$ be a natural number. If $T$ is a division ring, i.e. a (skew-) field, denote by $P^m(T)$, resp. $A^m(T)$, the $m$-dimensional projective, resp. affine, space over $T$, and by $p(u_1, \ldots, u_r)$ resp. $a(u_1, \ldots, u_r)$, the linear projective, resp. affine, variety genera-
ted by the points \( u_1, \ldots, u_r \) in \( P^m(T) \), resp. \( A^m(T) \). If \( u, v \in P^m(T) \) and \( u \neq v \) we write \( uv \) for \( p(u, v) \).

Let \( K, L \) be division rings. A function \( F : P^m(K) \to P^n(L) \), resp. \( F : A^m(K) \to P^n(L) \), is called a lineation if the images by \( F \) of any three collinear points are collinear. A lineation \( F \) is called full if there exist \( m+2 \) points in general position in \( P^m(K) \), resp. in \( A^m(K) \), carried by \( F \) into \( m+2 \) points in general position in \( P^n(L) \). Let \( \psi : K \to L \) be a place. If \( (v_0, \ldots, v_{m+1}) \in P^m(K) \), resp. \( (v'_0, \ldots, v'_{m+1}) \in P^n(L) \) are in general position, taking \( v = (v_0, \ldots, v_{m+1}) \) and \( v' = (v'_0, \ldots, v'_{m+1}) \) as reference frames for projective coordinate systems in \( P^m(K) \) and \( P^n(L) \), a lineation \( P^m(\psi; v, v') : P^m(K) \to P^n(L) \) is defined as follows: if \( [x^0, \ldots, x^m] \in P^m(K) \), take \( a \in K \) s.t. \( ax^0, \ldots, ax^m \) are in \( \text{dom } \psi \), but not all in \( \ker \psi \); then put

\[
(1.0) \quad P^m(\psi; v, v')([x^0, \ldots, x^m]) = [\psi(ax^0), \ldots, \psi(ax^m)].
\]

The definition is consistent ([8], Lemma 6 or [4], 2.6). We shall write \( P^m(\psi) \) instead of \( P^m(\psi; v, v') \) when no confusion can appear.

(1.1) Remark. Let \( \phi : K \to L \) be a place. In the above notation, if there exist \( i, j \in \{0, \ldots, m\} \), with \( i \neq j \) s.t. \( P^m(\phi; v, v') \) coincides with \( P^m(\phi; v_i, v_j) \) on \( v_i v_j \), then \( \psi = \phi \). Indeed put \( i = 0 \) and \( j = 1 \) and let \( a \in K \setminus \{0\} \). By definition \( P^m(\psi) ([1, a, 0, \ldots, 0]) = [1, \psi(a), 0, \ldots, 0] \) if \( \text{adom } \psi \) and \( P^m(\psi) ([a^{-1}, 1, 0, \ldots, 0]) = [0, 1, 0, \ldots, 0] \) if \( \text{adom } \psi \). The same equalities hold for \( P^m(\phi) \). Hence \( \text{dom } \psi = \text{dom } \phi \) and \( \psi = \phi \).

The aims of this paper are: to extend an affine full lineation to the projective envelope of its domain (3.5); to construct and study "bad" injective lineations (5.2-5.5) as a complementary case to the criterion 4.1 for an injective lineation to be full and surjective.

The extension of an affine full lineation to the projective envelope of its domain is based on: the plane case, solved in [5], 5.1 for almost all plane lineations with few exceptions; some geometric properties of affine full lineations, similar to those of the projective ones (3.2 and 3.4); the "coincidence" lemma 3.1. It would be interesting to solve this problem for a larger class of lineations and to classify all lineations \( F : A^m(R) \to A^m(R) \). These