USING L. S. PONTRYAGIN'S MAXIMUM PRINCIPLE IN MINIMUM-CRITICAL-SIZE AND MAXIMUM-POWER REACTOR PROBLEMS

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L. S. Pontryagin's maximum principle is used for solving problems in which it is necessary to find the minimum critical size of a reactor for a given power or to find the maximum power for a given critical size. Recently Pontryagin's maximum principle [1] has been successfully used both for determining the optimum transient conditions for reactors [2, 4] and for finding the optimum spatial arrangement for reactors with prescribed physical characteristics [1]. In the present study this principle is used in two other problems encountered in the theory of reactor design.

STATEMENT OF THE PROBLEM OF FINDING THE MINIMUM CRITICAL SIZE FOR A PRESCRIBED REACTOR POWER

It is assumed that the reactor power \( W \) is given and that the structural materials are distributed uniformly through the reactor. Resonance absorption and neutron absorption and multiplication during moderation are ignored. The uranium concentration \( U(z) \) is considered variable over the volume of the reactor within the limits:

\[
0 \leq U(z) \leq U_{\text{max}}.
\]  (1)

The power per unit of reactor volume, \( \rho(z) N(z)U(z) \), is limited:

\[
\rho = N(z) U(z) - D \leq 0,
\]  (2)

where \( N(z) \) is the thermal-neutron density at point \( z \) and \( D \) is a constant. It is required to find the distribution \( U(z) \) which will yield the minimum critical reactor size for a prescribed power \( W \) and under conditions (1) and (2). The problem is solved in a two-group approximation for a symmetric slab reactor. The initial equations describing the thermal-neutron density, \( [N(z)] \), and the moderated-neutron density \( [n(z)] \), have the usual form [6]

\[
\begin{align*}
\frac{d^2N}{dz^2} + \frac{1 + U(z)}{L_2^2} N &= -n; \\
\frac{d^2n}{dz^2} - \frac{n}{\tau} &= -\frac{\eta U}{\nu L_2^2} N,
\end{align*}
\]  (3)

where \( L_2^2 \) is the square of the diffusion length of the medium, where the structural materials are taken into account but the uranium is not; \( \tau \) is the square of the moderation length (the variation in \( \tau \) for different uranium concentrations is neglected); \( \eta \) is the effective number of neutrons produced in fission.

PONTRYAGIN'S METHOD

In order to use the mathematical theory of optimal processes [1], we write (3) in the form of four first-order equations, introducing the notation \( x^{(1)} = N; x^{(2)} = dN/dz; x^{(3)} = n; x^{(4)} = dn/dz \) and adding an equation for \( x^{(5)} \), making use of the fact that the reactor power is specified and is equal to \( W = \int_0^H N(x)U(x)dx \) (where \( H \) is the desired half-width of the reactor). As a result, we obtain the following system of equations:
\[
\begin{aligned}
\frac{dx(1)}{dz} &= x(2) \equiv f(1); \\
\frac{dx(2)}{dz} &= \frac{1+U}{L_4} x(1) - x(3) \equiv f(2); \\
\frac{dx(3)}{dz} &= x(4) \equiv f(3); \\
\frac{dx(4)}{dz} &= \frac{x(3)}{U} - \eta \frac{U}{L_4} x(1) \equiv f(4); \\
\frac{dx(5)}{dz} &= \eta x(4) \equiv f(5).
\end{aligned}
\] (4)

The functions \(x^{(i)}\) satisfy the following boundary conditions:
\[
\begin{aligned}
x^{(1)}(H) &= x^{(3)}(H) = 0; \quad x^{(2)}(0) = x^{(1)}(0) = 0; \quad x^{(5)}(0) = 0; \quad x^{(4)}(H) = W.
\end{aligned}
\] (5)

The Hamiltonian of the system (4) is formed according to the rule [1]:
\[
\mathcal{H} = \sum_{i=1}^{5} \psi_i f^{(i)} = \chi + U \psi;
\] (6)

where the auxiliary functions \(\psi_i\) satisfy the equations
\[
\frac{\partial \psi_i}{\partial z} = -\frac{\partial \mathcal{H}}{\partial x^{(i)}} + \lambda \frac{\partial \psi}{\partial x^{(i)}},
\] (7)

and the function \(\psi\) is given by formula (2). The function \(\lambda\) is defined as follows: if \(p < 0\), then \(\lambda \equiv 0\); if \(p = 0\), then \(\lambda\) is defined by the condition
\[
\frac{\partial \mathcal{H}}{\partial \psi} = \lambda \frac{\partial \psi}{\partial U}.
\] (8)

Taking into account the boundary conditions (5) for the functions \(x^{(i)}\) and the transversality conditions for the functions \(\psi_i\), we obtain the following boundary conditions:
\[
\psi_1(0) = \psi_3(0) = 0; \quad \psi_2(H) = \psi_4(0) = 0.
\] (9)

Pontryagin's maximum principle* requires that in order to find the optimum distribution \(U(z)\), we must find a continuous and not identically equal to zero vector function \(\psi\) (with components \(\psi_i\), \(i = 1, \ldots, 5\)) such that, first of all, the Hamiltonian \(\mathcal{H}\) as a function of the independent variable \(U\) reaches a supremum \(\mathcal{M} = \sup_{U} \mathcal{H} \equiv M\) everywhere in the region \(0 \leq z \leq H\); secondly, the supremum of the Hamiltonian \(\mathcal{H}\) is a constant positive number; and thirdly, the condition \(\lambda \equiv 0\) is satisfied in the zone where \(p = 0\), if such a zone exists.

**ADMISSIBLE TYPES OF CONTROL**

Pontryagin's maximum principle enables us at once to determine the admissible types of control; i.e., to determine the types of zones, with known control functions of which the reactor can consist. It is evident that in the present problem the Hamiltonian as a function of the independent variable \(U\) will attain a supremum in the following cases:

1. \(U(z) = U_{\text{max}}\), if \(\varphi > 0\).
2. \(U(z) = 0\), if \(\varphi(z) < 0\).
3. \(U(z) = U_0(z)\).

* According to the terminology of [1], this problem of the minimum critical size belongs to the class of fast-response problems.