USING L. S. PONTRYAGIN'S MAXIMUM PRINCIPLE IN MINIMUM-CRITICAL-SIZE AND MAXIMUM-POWER REACTOR PROBLEMS

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L. S. Pontryagin's maximum principle is used for solving problems in which it is necessary to find the minimum critical size of a reactor for a given power or to find the maximum power for a given critical size. Recently Pontryagin's maximum principle [1] has been successfully used both for determining the optimum transient conditions for reactors [2, 4] and for finding the optimum spatial arrangement for reactors with prescribed physical characteristics [1]. In the present study this principle is used in two other problems encountered in the theory of reactor design.

STATEMENT OF THE PROBLEM OF FINDING THE MINIMUM CRITICAL SIZE FOR A PRESCRIBED REACTOR POWER

It is assumed that the reactor power $W$ is given and that the structural materials are distributed uniformly through the reactor. Resonance absorption and neutron absorption and multiplication during moderation are ignored. The uranium concentration $U(z)$ is considered variable over the volume of the reactor within the limits:

$$0 \leq U(z) \leq U_{\text{max}}. \quad (1)$$

The power per unit of reactor volume, $p(z) = N(z)U(z)$, is limited:

$$p = N(z)U(z) - D \leq 0, \quad (2)$$

where $N(z)$ is the thermal-neutron density at point $z$ and $D$ is a constant. It is required to find the distribution $U(z)$ which will yield the minimum critical reactor size for a prescribed power $W$ and under conditions (1) and (2). The problem is solved in a two-group approximation for a symmetric slab reactor. The initial equations describing the thermal-neutron density, $[N(z)]$, and the moderated-neutron density $[n(z)]$, have the usual form [6]

$$\begin{align*}
\frac{d^2N}{dz^2} - \frac{1 + U(z)}{L_3^2} N &= -n; \\
\frac{d^2n}{dz^2} - \frac{n}{\tau} &= -\frac{\eta U}{\nu L_5^2} N,
\end{align*} \quad (3)$$

where $L_3^2$ is the square of the diffusion length of the medium, where the structural materials are taken into account but the uranium is not; $\tau$ is the square of the moderation length (the variation in $\tau$ for different uranium concentrations is neglected); $\eta$ is the effective number of neutrons produced in fission.

PONTRYAGIN'S METHOD

In order to use the mathematical theory of optimal processes [1], we write (3) in the form of four first-order equations, introducing the notation $x(1) = N; x(2) = dN/dz; x(3) = n; x(4) = dn/dz$ and adding an equation for $x(5)$, making use of the fact that the reactor power is specified and is equal to $W = \int_0^H N(x)U(x)dx$ (where $H$ is the desired half-width of the reactor). As a result, we obtain the following system of equations:

\[
\begin{align*}
\frac{dx^{(1)}}{dz} &= x^{(2)} = f^{(1)}; \\
\frac{dx^{(2)}}{dz} &= \frac{1+U_{k}}{L_{k}} x^{(1)} - x^{(3)} = f^{(2)}; \\
\frac{dx^{(3)}}{dz} &= x^{(1)} = f^{(3)}; \\
\frac{dx^{(4)}}{dz} &= \frac{x^{(3)}}{\tau} - \frac{\eta_{U}}{\tau_{L}} x^{(1)} = f^{(4)}; \\
\frac{dx^{(5)}}{dz} &= U x^{(1)} = f^{(5)}.
\end{align*}
\]

The functions \( x^{(i)} \) satisfy the following boundary conditions:
\[
x^{(1)}(H) = x^{(3)}(H) = 0; \quad x^{(2)}(0) = x^{(1)}(0) = 0; \quad x^{(5)}(0) = 0; \quad x^{(5)}(H) = W.
\]

The Hamiltonian of the system (4) is formed according to the rule [1]:
\[
\mathcal{H} = \sum_{i=1}^{5} \psi_{i} f^{(i)} = x + U \psi;
\]
\[
\lambda = \psi_{1} x^{(2)} + \frac{1}{L_{k}} \psi_{1} x^{(1)} - \psi_{2} x^{(3)} + \psi_{3} x^{(4)} + \psi_{4} x^{(5)}; \\
\psi = x^{(1)} \left[ \frac{1}{L_{k}} \left( \psi_{2} - \frac{\eta_{U}}{\tau} \psi_{1} \right) + \psi_{5} \right],
\]
where the auxiliary functions \( \psi_{i} \) satisfy the equations
\[
\frac{d\psi_{i}}{dz} = -\frac{\partial \mathcal{H}}{\partial x^{(i)}} + \lambda \frac{\partial \nu}{\partial x^{(i)}},
\]
and the function \( \nu \) is given by formula (2). The function \( \lambda \) is defined as follows: if \( \nu < 0 \), then \( \lambda = 0 \); if \( \nu > 0 \), then \( \lambda \) is defined by the condition
\[
\frac{\partial \mathcal{H}}{\partial \nu} = \lambda \frac{\partial \nu}{\partial \nu}.
\]

Taking into account the boundary conditions (5) for the functions \( x^{(i)} \) and the transversality conditions for the functions \( \psi_{i} \), we obtain the following boundary conditions:
\[
\psi_{1}(0) = \psi_{3}(0) = 0; \quad \psi_{2}(H) = \psi_{4}(H) = 0.
\]

Pontryagin’s maximum principle* requires that in order to find the optimum distribution \( U(z) \), we must find a continuous and not identically equal to zero vector function \( \psi \) (with components \( \psi_{i} \), \( i = 1, \ldots, 5 \)) such that, first of all, the Hamiltonian \( \mathcal{H} \) as a function of the independent variable \( U \) reaches a supremum \( \mathcal{H} = \sup_{U} \mathcal{H} = M \) everywhere in the region \( 0 \leq z \leq H \); secondly, the supremum of the Hamiltonian \( \mathcal{H} \) is a constant positive number; and thirdly, the condition \( \lambda = 0 \) is satisfied in the zone where \( \nu = 0 \), if such a zone exists.

**ADMISSIBLE TYPES OF CONTROL**

Pontryagin’s maximum principle enables us at once to determine the admissible types of control; i.e., to determine the types of zones, with known control functions of which the reactor can consist. It is evident that in the present problem the Hamiltonian as a function of the independent variable \( U \) will attain a supremum in the following cases:

1. \( U(z) = U_{\text{max}} \), if \( \nu > 0 \).
2. \( U(z) = 0 \), if \( \nu(z) < 0 \).
3. \( U(z) = U_{0}(z) \).

* According to the terminology of [1], this problem of the minimum critical size belongs to the class of fast-response problems.