LINEAR PROGRAMMING BOUNDS FOR CODES IN INFINITE PROJECTIVE SPACES

Peter Boyvalenkov, Danyo Danev, Maria Mitradjieva

We develop a technique for improving the universal linear programming bounds on the cardinality and the minimum distance of codes in projective spaces \( \mathbb{F}P^{n-1} \). We firstly investigate test functions \( P_j(m, n, s) \) having the property that \( P_j(m, n, s) < 0 \) for some \( j \) if and only if the corresponding universal linear programming bound can be further improved by linear programming. Then we describe a method for improving the universal bounds. We also investigate the possibilities for attaining the first universal bounds.

1 INTRODUCTION

We consider codes in the projective spaces \( \mathbb{F}P^{n-1} \) of lines through the origin in \( \mathbb{F}^n \), where \( \mathbb{F} \) denotes the real numbers \( \mathbb{R} \), the complex numbers \( \mathbb{C} \), or the quaternions \( \mathbb{H} \). Together with the Euclidean spheres \( S^{n-1} \) they constitute connected compact symmetric spaces of rank 1. For a detailed description of these spaces we refer to [9, 15, 16]. More general references are [8, 19, 21]. Set \( m = \dim(\mathbb{F} : \mathbb{R}) \).

The points \( x = (x_1, x_2, \ldots, x_n) \in \mathbb{F}^n \) and \( y = (y_1, y_2, \ldots, y_n) \in \mathbb{F}^n \) are said to be equivalent if there exists \( \lambda \in \mathbb{F} \setminus \{0\} \) such that \( x_i = \lambda y_i \) for all \( i = 1, 2, \ldots, n \). Then the space \( \mathbb{F}P^{n-1} \) consists of all equivalence classes of nonzero points from \( \mathbb{F}^n \). For \( X, Y \in \mathbb{F}P^{n-1} \) we define the function

\[
\alpha(X, Y) = \frac{|(x, y)|}{\sqrt{(x, x)(y, y)}},
\]

where \( x \in X, y \in Y \) and \( (x, y) = \sum_{i=1}^{n} x_i y_i^* \) is the usual inner product in \( \mathbb{F}^n \). The angle \( \varphi = \varphi(X, Y) \in [0, \pi/2] \) such that \( \cos \varphi(X, Y) = \alpha(X, Y) \) is called angle between the lines \( X \) and \( Y \) (it is the angle in the usual sense when \( \mathbb{F} = \mathbb{R} \)). The metric in \( \mathbb{F}P^{n-1} \) is defined by

\[
d(X, Y) = \sqrt{1 - \alpha(X, Y)} = \sqrt{1 - \cos \varphi(X, Y)} = \sqrt{2 \sin \frac{\varphi(X, Y)}{2}} \in [0, 1]
\]
and the inner product is \((X, Y) = (2(1 - d^2(X, Y))^2 - 1 = 2a^2(X, Y) - 1 \in [-1, 1])

Any finite set \(W \subset \mathbb{F}^{n-1}\) is called a code. If \(W \subset \mathbb{F}^{n-1}\) is a code, then the quantities \(d(W) = \min\{d(X, Y) | X, Y \in W, X \neq Y\} \in (0, 1]\) and \(s(W) = 2(1 - d^2(W))^2 - 1 = \max\{(X, Y) | X, Y \in W, X \neq Y\} \in [-1, 1]\) are said to be the minimum distance of \(W\) and the maximal inner product of \(W\) respectively. If \(|W| = M\) and \(s(W) = s\), the code \(W\) is said to be an \((m, n, M, s)\) code.

The maximal possible size among all codes \(W \subset \mathbb{F}^{n-1}\) with prescribed \(s = s(W)\) is \(A(m, n, s) = \max\{|W| : s(W) = s\}\). The minimum possible maximal inner product among all codes in \(\mathbb{F}^{n-1}\) of fixed cardinality \(M\) is denoted by \(s(m, n, M)\), i.e. \(s(m, n, M) = \min\{s(W) : |W| = M\}\).

Every space \(\mathbb{F}^{n-1}\) is naturally connected [9, 13] with its zonal spherical functions which are normalized Jacobi polynomials \(Q_k^{(\alpha, \beta)}(t)\) \([1, \text{Chapter 22}]\), where

\[
Q_k^{(\alpha, \beta)}(t) = \frac{1}{2^k \binom{k + \alpha}{k} \binom{k + \beta}{k - i}} \sum_{i=0}^{k} \binom{k + \alpha}{i} \binom{k + \beta}{k - i} (t + 1)^i (t - 1)^{k-i} = \sum_{i=0}^{k} a_{k,i} t^i, \quad k \geq 0.
\]

The best lower bounds on \(s(m, n, M)\) and upper bounds on \(A(m, n, s)\) are as usually in the coding theory these obtained by linear programming [6, 7, 13, 14, 20, 17]. The following theorem [7, 12] gives the so-called linear programming bound for codes in \(\mathbb{F}^{n-1}\).

**THEOREM 1.1.** ([7, 12]) Let \(m, n\) and \(s\) be fixed and \(f(t)\) be a real polynomial such that (A1) \(f(t) \leq 0\) for \(-1 \leq t \leq s\), and (A2) The coefficients in the Jacobi expansion \(f(t) = \sum_{i=0}^{k} f_i P_i^{(\alpha, \beta)}(t)\) (\(\alpha\) and \(\beta\) are given by (1)) satisfy \(f_0 > 0, f_i \geq 0\) for \(i = 1, \ldots, k\). Then \(A(m, n, s) \leq f(1)/f_0\).

Upper bounds on \(A(m, n, s)\) were firstly obtained by Sidelnikov [17] and Welch [20]. At present, the best known universal upper bound on \(A(m, n, s)\) is the Levenshtein bound [13, 14, 15]:

\[
A(m, n, s) \leq \begin{cases} 
L_{2k-1}(m, n, s) = \left(1 - \frac{Q_{k-1}^{(\alpha+1, \beta)}(s)}{Q_k^{(\alpha, \beta)}(s)}\right) R_{k-1} & \text{for } t_{k-1}^{1,1} \leq s \leq t_{k-1}^{1,0}, \\
L_{2k}(m, n, s) = \left(1 - \frac{Q_k^{(\alpha+1, \beta)}(s)}{Q_k^{(\alpha+1, \beta+1)}(s)}\right) R_{k} & \text{for } t_{k}^{1,0} \leq s \leq t_{k}^{1,1},
\end{cases}
\]

where \(t_{k-1}^{1,1}\) and \(t_{k}^{1,0}\) are the greatest zeros of the Jacobi polynomials \(Q_i^{(\alpha+1, \beta+1)}(t)\) and \(Q_i^{(\alpha+1, \beta)}(t)\) respectively, and

\[
R_k = \binom{k + \alpha + \beta + 1}{k} \binom{k + \alpha + 1}{k} / \binom{k + \beta}{k}.
\]