A REGULARIZATION OF THE THREE-BODY PROBLEM

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Abstract. Let \( r_1, r_2, r_3 \) be arbitrary coordinates of the non-zero interacting mass-points \( m_1, m_2, m_3 \) and define the distances \( R_1 = |r_1 - r_3|, R_2 = |r_2 - r_3|, R = |r_1 - r_2| \). An eight-dimensional regularization of the general three-body problem is given which is based on Kustaanheimo-Stiefel regularization of a single binary and possesses the properties:

(i) The equations of motion are regular for the two-body collisions \( R_1 \rightarrow 0 \) or \( R_2 \rightarrow 0 \).
(ii) Provided that \( R \gtrsim R_1 \) or \( R \gtrsim R_2 \), the equations of motion are numerically well behaved for close triple encounters.

Although the requirement \( R \gtrsim \min (R_1, R_2) \) may involve occasional transformations to physical variables in order to re-label the particles, all integrations are performed in regularized variables. Numerical comparisons with the standard Kustaanheimo-Stiefel regularization show that the new method gives improved accuracy per integration step at no extra computing time for a variety of examples. In addition, time reversal tests indicate that critical triple encounters may now be studied with confidence.

The Hamiltonian formulation has been generalized to include the case of perturbed three-body motions and it is anticipated that this procedure will lead to further improvements of \( N \)-body calculations.

1. Introduction

According to Whittaker (1904), the general three-body problem is the most celebrated problem in dynamical astronomy. Considerable theoretical and numerical efforts have been devoted to the study of this fascinating subject but, for reasons of simplicity, most investigations have been concerned with the corresponding circular restricted problem. However, the numerical approach is more versatile, permitting arbitrary configurations to be considered. The rapid advance of high-speed computers has therefore increased the interest in the general three-body problem. At the same time, the numerical challenge has led to the introduction of many new techniques. In the first instance, classical perturbation methods which are based on the concept of dominant two-body motion have been used to calculate the orbits of planets and comets. Nevertheless, direct integration methods which are not restricted to small perturbations continue to enjoy considerable popularity because of their simplicity. On the other hand, the effectiveness of direct calculations is reduced during close approaches between the interacting bodies. Such events occur frequently in three-body systems; consequently, the accuracy of the solutions cannot be guaranteed in many cases. General considerations suggest that an ideal procedure should combine the greater flexibility of direct calculations with the advantage of using a perturbed two-body description, which gives regular resolutions in the neighbourhood of attracting centres.

The first example of regularization is due to Euler (1765) who employed a simple time transformation to eliminate the collision singularity of two particles moving on a

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straight line. In a classical paper, Levi-Civita (1903) showed that one of the singularities of the planar three-body problem can be removed by introducing new coordinates as well as the time transformation of Euler. The resulting equations of motion are regular if only one distance tends to zero, and the corresponding solutions are then well behaved. However, the principle of Levi-Civita regularization is based on complex variables, which cannot be extended to three dimensions. For a long time this fundamental stumbling block prevented further progress, until Kustaanheimo and Stiefel (1965), by an elegant four-dimensional generalization of the Levi-Civita procedure, succeeded in regularizing the general two-body problem with arbitrary external perturbations. This remarkable achievement, subsequently referred to as standard KS regularization, has stimulated further significant advances and has also done much to popularize the principles of regularization. It is indicative of the numerical effectiveness that the regularized solutions of any dominant two-body motion are improved with respect to direct integration methods.

The history of three-body regularization dates back to the celebrated works of Poincaré (1907) and Sundman (1912). In fact, Sundman solved the general problem in principle, excepting the case of triple collision, by introducing two time transformations. Unfortunately, the solutions are specified by infinite power series which do not reveal the behaviour of the motions. It is also doubtful whether the three-body regularization proposed by Lemaitre (1955) is practically useful in the general case. Consequently, the introduction of the KS theory can properly be said to have set the stage for a renewed attack on the dynamical summit represented by the general three-body problem. This challenge divides naturally into two parts:

(i) Regularization of all two-body collisions by one ‘global’ transformation.
(ii) Improved treatment of close triple encounters.

Examples of transformations satisfying the first requirement in the planar restricted three-body problem are due to Thiele (1896), Burrau (1906), Birkhoff (1915) and Lemaitre (1955). An equivalent three-dimensional formulation of Birkhoff’s regularization has been given by Stiefel and Waldvogel (1965); in all cases the body of infinitesimal mass can have repeated collisions with either of the two primaries without change of variables. It may be noted that, until quite recently, alternative methods satisfying the second requirement were not available. A new milestone was therefore reached when Waldvogel (1972) obtained a global regularization of the planar three-body problem with arbitrary masses. The corresponding equations of motion are symmetrical with respect to the individual mass-points and have the desirable property of satisfying both the requirements above.

Although current regularization theory is still not able to deal with the most general problem in a practical way, recent advances have been encouraging and permit new avenues to be explored. The present approach is based on familiar ideas of Hamiltonian dynamics in the extended phase space, and employs the six centre of mass integrals to reduce the order of the differential equations which describe the particle motions. In the following derivation we rely heavily on the classical canonical treatment of Whittaker (1904), as well as the modern discourse on regularization by