SOME MANIFOLDS OF PERIODIC ORBITS IN THE RESTRICTED THREE-BODY PROBLEM

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ABSTRACT. In the present paper we give some numerical results about natural families of periodic orbits, which emanate from limiting orbits around the equilateral equilibrium points of the Restricted Three-Body Problem, when the mass ratio is greater than Routh's critical one.

1. INTRODUCTION

Let \( m_1, m_2 \) be the masses of the primaries normalized in such a way that \( m_1 = \mu, m_2 = 1 - \mu, \mu \in [0, 1] \). Units of length and time are chosen in order to have one unit of distance between the primaries and a mean motion equal to one.

In a synodical system of coordinates \((x, y)\) the two primaries are fixed at \((1 - \mu, 0)\) and \((-\mu, 0)\), respectively. The equations of motion are (see [13])

\[
\begin{align*}
\ddot{x} - 2\dot{y} &= \Omega_x, \\
\ddot{y} + 2\dot{x} &= \Omega_y
\end{align*}
\]

where

\[
\Omega(x, y) = \frac{1}{2} \left( \mu \frac{r_2^2}{r_1} + (1 - \mu) \frac{r_1^2}{r_2} \right) + \frac{\mu}{r_1} + \frac{1 - \mu}{r_2}
\]

and

\[
\begin{align*}
r_1^2 &= (x - 1 + \mu)^2 + y^2, \\
r_2^2 &= (x + \mu)^2 + y^2.
\end{align*}
\]

System (1) has the Jacobian integral

\[
C = 2\Omega(x, y) - (x^2 + y^2).
\]
We consider the equilibrium point $L_4$, i.e.

$$x = 1/2 - \nu, \quad y = \sqrt{3}/2.$$ 

In connection with this equilateral point, there exists a critical mass ratio

$$\nu_1 = \frac{1}{2} \left[ 1 - \frac{1}{9} (69)^{1/2} \right] = 0.03852...$$

which has the following properties (see [13]):

(i) All four characteristic exponents of the equilibrium are distinct if and only if $\nu \neq \nu_1, 1 - \nu_1$.

(ii) For $\nu < \nu_1$ they are of the form $\pm \imath n_s$, where the real numbers $n_s$ and $n_z$ satisfy the inequalities

$$0 < n_z < 1/\sqrt{2} < n_s < 1$$

they are thus of the linearly stable type. Similar for $\nu > 1 - \nu_1$.

(iii) For $1 - \nu_1 > \nu > \nu_1$ the characteristic exponents are of the form $\alpha \pm \imath \beta$, where the real numbers $\alpha$ and $\beta$ are both different from zero, they are thus of the unstable type.

Lyapunov's theorem (see [11]) establishes that:

(a) For any $\nu \in (0, \nu_1)$, there emanates from $L_4$ a family $L_4^S(\nu)$ of short period orbits which depends analytically on a real parameter $\epsilon$; when $\epsilon$ goes to zero, the periodic orbit vanishes at the point $L_4$ itself, and its period tends to $2\pi/n_s$.

(b) For any $\nu \in (0, \nu_1)$, except at the critical mass ratios

$$\nu_k = \frac{1}{2} \left[ 1 - \frac{16k^2}{1 - 27(k^2 + 1)^2} \right]^{1/2}, \quad k = 1, 2, ...$$

there exists another family of periodic orbits around $L_4$ which depends analytically on a real parameter $\epsilon$; when $\epsilon$ goes to zero, the periodic orbit vanishes at the point $L_4$ itself, but this time, its period tends to $2\pi/n_k$. Hence in view of (2), the family is called the family of long period orbits at $L_4$, $L_4^L(\nu)$.

For $\nu$ slightly greater than $\nu_1$, Brown [1] analyzed the equilibrium configuration up to the third order. His conclusions were the following:

(i) There still exist two families of periodic orbits at $L_4$ and both families depend analytically on a real parameter $\epsilon$.

(ii) This orbital parameter admits a strictly positive lower bound $\delta$ the same for both families. $\delta$ is an analytical function of the mass ratio $\nu$; when $\epsilon$ goes to zero, both families close to a common periodic orbit, which is called the limiting orbit.

(iii) The limiting orbit is properly periodic, the period being equal to $2\pi/\sqrt{2}$ whatever the mass ratio may be. For $\alpha > 0$, the limiting orbit stays at finite distance from $L_4$. When $\nu$ tends to the critical value $\nu_1$, $\delta$ goes to zero and the limiting orbit vanishes at $L_4$. 