THREE-DIMENSIONAL PERIODIC SOLUTIONS AROUND EQUILIBRIUM POINTS IN HILL'S PROBLEM

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ABSTRACT. The three-dimensional periodic solutions originating at the equilibrium points of Hill's limiting case of the Restricted Three Body Problem, are studied. Fourth-order parametric expansions by the Lindstedt-Poincaré method are constructed for them. The two equilibrium points of the problem give rise to two exactly symmetrical families of three-dimensional periodic solutions. The family $H_{L2}$ originating at $L_2$ is continued numerically and is found to extend to infinity. The family originating at $L_1$ behaves in exactly the same way and is not presented. All orbits of the two families are unstable.

1. INTRODUCTION

In the circular restricted problem of three bodies there are five points of equilibrium, the collinear equilibrium points $L_1, L_2, L_3$ and the triangular equilibrium points $L_4$ and $L_5$.

In Hill's limiting case of the restricted problem when the mass ratio of the primaries is vanishingly small, there are two points of equilibrium named $L_1$ and $L_2$. The third collinear point $L_3$ and the two triangular points $L_4$ and $L_5$ have been removed to infinity by the limiting process. With the origin of the coordinate system at the small body the coordinates of the points $L_1, L_2$ are $(t^{2/3}, 0, 0)$.

In the planar case of this problem the families $a$ and $c$ of periodic solutions emanating from the above two equilibrium points have been computed by Hénon (1969).

In this paper we study the three dimensional periodic motions about the positions of equilibrium of the Hill problem. Periodic solutions are approximated via a fourth order parametric expansion with respect to an orbital parameter. These solutions are continued numerically to a family of periodic orbits which are symmetrical w.r.t. the Oxz-plane and the Ox-axis. The size of these orbits increases along the family tending to infinity. This family is found to bifurcate with the family $a_{20}$ of periodic
2. EQUATIONS - POINTS OF EQUILIBRIUM

In a rotating dimensionless cartesian coordinate system with origin on the vanishingly small body the differential equations of motion for the problem of Hill are:

\[
\ddot{X} - 2\dot{Y} = 3X - \frac{X}{R^3}, \quad (1a)
\]
\[
\ddot{Y} + 2\dot{X} = -\frac{Y}{R^3}, \quad (1b)
\]
\[
\ddot{Z} = Z(-1 - \frac{1}{R^3}) \quad (1c)
\]

where \((X, Y, Z)\) are the coordinates of the 'third' (in reality - 'second') body, and \(R^2 = X^2 + Y^2 + Z^2\).

These equations admit the integral of Jacobi:

\[
C = 3X^2 + 2\frac{\dot{Z}}{R} - \dot{X}^2 - \dot{Y}^2 - \dot{Z}^2. \quad (2)
\]

For details see e.g. Szebehely (1967, p. 602).

The two collinear equilibrium points (both 'saddle' points) are:

\[
L_1(-3^{-1/3}, 0, 0) \quad \text{and} \quad L_2(3^{-1/3}, 0, 0).
\]

If \(X_0\) denotes the position on the Ox axis of \(L_1\) or \(L_2\), then by the transformation

\[
X = X_0 + \xi, \quad Y = \eta, \quad Z = \zeta, \quad (3)
\]

Equations (1) become

\[
\ddot{\xi} - 2\dot{\eta} = 3(X_0 + \xi) - \frac{X_0 + \xi}{R^3}, \quad (4a)
\]
\[
\ddot{\eta} + 2\dot{\xi} = -\frac{\eta}{R^3}, \quad (4b)
\]
\[
\ddot{\zeta} = \zeta\left(-1 - \frac{1}{R^3}\right) \quad (4c)
\]

with

\[
R^2 = X_0^2 \left[1 + \frac{\xi}{X_0} + \left(\frac{\xi}{X_0}\right)^2 + \left(\frac{\eta}{X_0}\right)^2 + \left(\frac{\zeta}{X_0}\right)^2\right]. \quad (5)
\]

After linearization the following 'variational' equations are obtained:

\[
\ddot{\xi} - 2\dot{\eta} = 9\xi, \quad (6a)
\]
\[
\ddot{\eta} + 2\dot{\xi} = -3\eta, \quad (6b)
\]
\[
\ddot{\zeta} = -4\zeta. \quad (6c)
\]